A note on causal relations and twistor space

Let $\mathbf{M}$ represent Minkowski space, and $\mathbf{PN^I}$ represent projective null twistor space without $\mathbf{I}$, the twistor line at infinity, regarded as a subset of $\mathbf{CP^3}$.

Lemma *If $x, y \in \mathbf{M}$, then $x \in J(y)$ if and only if there is a null curve in $\mathbf{M}$ joining x to y. In addition, $x \in I(y)$ if and only if this curve is not a geodesic.*
Proof Consider any null curve starting at x. This stays in $J(x)$ and enters $I(x)$ iff it is not a geodesic. Any point in $I(x)$ may be carried to any other by a Poincaré transformation fixing x. Such transformations do not alter the null or geodesic properties of a curve. □

If $\gamma : [0,1] \rightarrow \mathbf{M}$ is a smooth curve, denote by $\Gamma(t)$ the sky in $\mathbf{PN^I}$ of $\gamma(t)$. The curve then gives a ruled surface in $\mathbf{PN^I}$ denoted by $\tilde{\Gamma}$ and defined by $\{\Gamma(t) : t \in [0,1]\}$

Lemma *The surface $\tilde{\Gamma}$ is a developable if and only if γ is null.*
Proof A developable may be characterised as a ruled surface whose infinitesimally separated generators intersect (see, for example, Semple and Roth, *Introduction to algebraic geometry*, pp 255 ff). Now, neighbouring generators of $\tilde{\Gamma}$ are the skies of neighbouring points on γ and hence correspond to null separated points; they therefore intersect. □

Using these two, and observing that $\tilde{\Gamma}$ has self intersections only when two points of γ are null separated, one can show that the following two results hold.

Proposition *$x \in I(y)$ if and only if $X \cup Y$ is the boundary of a developable with no self intersections.* □

Corollary *Let Σ be a ruled surface in $\mathbf{PN^I}$ with boundary $X \cup Y$. Then $x \in I(y)$ if and only if Σ can be deformed into a developable.* □

From these results we see that the causal nature of the interval between two points in $\mathbf{M}$ admits of a fairly direct interpretation in $\mathbf{PN^I}$ in terms of its projective geometry. Note that it is important that we do not use $\mathbf{PN}$, as this would allow us to find a (non self-intersecting) developable with boundary $X \cup Y$ for any pair of points not lying on a single null geodesic.

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