A note on causal relations and twistor space

Let $M$ represent Minkowski space, and $PN^1$ represent projective null twistor space without $I$, the twistor line at infinity, regarded as a subset of $CP^3$.

Lemma If $x, y \in M$, then $x \in J(y)$ if and only if there is a null curve in $M$ joining $x$ to $y$. In addition, $x \in I(y)$ if and only if this curve is not a geodesic.

Proof Consider any null curve starting at $x$. This stays in $J(x)$ and enters $I(x)$ iff it is not a geodesic. Any point in $I(x)$ may be carried to any other by a Poincaré transformation fixing $x$. Such transformations do not alter the null or geodesic properties of a curve. □

If $\gamma : [0, 1] \to M$ is a smooth curve, denote by $\Gamma(t)$ the sky in $PN^1$ of $\gamma(t)$. The curve then gives a ruled surface in $PN^1$ denoted by $\tilde{\Gamma}$ and defined by $\{\Gamma(t) : t \in [0, 1]\}$

Lemma The surface $\tilde{\Gamma}$ is a developable if and only if $\gamma$ is null.

Proof A developable may be characterised as a ruled surface whose infinitesimally separated generators intersect (see, for example, Semple and Roth, *Introduction to algebraic geometry*, pp 255 ff). Now, neighbouring generators of $\tilde{\Gamma}$ are the skies of neighbouring points on $\gamma$ and hence correspond to null separated points; they therefore intersect. □

Using these two, and observing that $\tilde{\Gamma}$ has self intersections only when two points of $\gamma$ are null separated, one can show that the following two results hold.

Proposition $x \in I(y)$ if and only if $X \cup Y$ is the boundary of a developable with no self intersections. □

Corollary Let $\Sigma$ be a ruled surface in $PN^1$ with boundary $X \cup Y$. Then $x \in I(y)$ if and only if $\Sigma$ can be deformed into a developable. □

From these results we see that the causal nature of the interval between two points in $M$ admits of a fairly direct interpretation in $PN^1$ in terms of its projective geometry. Note that it is important that we do not use $PN$, as this would allow us to find a (non self-intersecting) developable with boundary $X \cup Y$ for any pair of points not lying on a single null geodesic.

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