Harmonic morphisms and mini-twistor space.

A harmonic morphism is a map $\varphi : M \to N$ of Riemannian manifolds M, N with the following property: *$f : N \to R$ is harmonic iff $\varphi \circ f : M \to R$ is.* As a concrete example, take M to be $\mathbf{R}^3$ with coordinates x, y, z and N to be $\mathbf{R}^2$ with coordinates u, v. The map φ is defined by giving $u(x,y,z)$, $v(x,y,z)$ satisfying

$$\nabla^2 u = \nabla^2 v = 0 = \nabla u \cdot \nabla v \; ; \; |\nabla u|^2 = |\nabla v|^2 \qquad (1).$$

In this case Baird and Wood [**BW**] find that φ is locally defined by a holomorphic curve in **T**P1, the tangent bundle of the complex projective line. They go on to use this fact to classify globally defined harmonic morphisms in this case, and also in the cases $\mathbf{S}^3 \to$ surface and $\mathbf{H}^3 \to$ surface.

Since **T**P1 is the mini-twistor space of $\mathbf{R}^3$ it is natural to wonder what, if anything, is the relation to twistor theory of this property of φ. In the case when dim M = 3, dim N = 2, the inverse images of points of N give curves in M. One purpose of this note is to observe that

the defining property of harmonic morphisms is equivalent to the condition that this congruence of curves be a geodesic and shear-free congruence.

Now **T**P1 is the space of geodesics of the flat metric on $\mathbf{R}^3$ and so a congruence of geodesics corresponds to a 2-real parameter surface in **T**P1. As one might anticipate from the Kerr theorem, there is a mini-Kerr theorem that

this surface is a holomorphic curve iff the congruence is shear-free.

In particular, this leads to an explicit formula for such congruences: if the generator is

$$L = \frac{1-\alpha\alpha}{1+\alpha\alpha}\frac{\partial}{\partial z} + \frac{\alpha+\alpha}{1+\alpha\alpha}\frac{\partial}{\partial x} - \frac{i(\alpha-\alpha)}{1+\alpha\alpha}\frac{\partial}{\partial y}$$

then $\alpha(x,y,z)$ is given implicitly by

$$f(x(1-\alpha^2) + iy(1+\alpha^2) + 2\alpha z, \alpha) = 0 \quad \text{or in spinors} \quad F(x^{AB}\alpha_A\alpha_B, \alpha_C) = 0$$

for arbitrary holomorphic f or holomorphic and homogeneous F (a formula similar to this is in [**BW**].

As Baird and Wood remark, to find solutions of (1) was set as a problem by Jacobi. This now falls into the class of non-linear differential-geometric problems solvable by twistor theory.

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BW Baird and Wood 1988 Math. Ann. **280** 579-603
see also Baird 1987 Ann. Inst. Fourier, Grenoble 37 135-173
Baird and Wood *Harmonic morphisms and conformal foliations by geodesics of three-dimensional space-forms* University of Melbourne Department of Mathematics Research Report no.2-1989

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