Twistors and SU(3) monopoles

Hitchin [1] has shown that SU(2)-monopoles of charge k on $\mathbb{R}^3$ are equivalent to algebraic curves (spectral curves) of genus $(k-1)^2$, satisfying certain constraints, lying in the minitwistor space $T\mathbb{P}^1$. Now $T\mathbb{P}^1 = \{(\underline{u}, \underline{v}): \underline{u}, \underline{v} \in \mathbb{R}^3; \|\underline{u}\| = 1, \underline{u}.\underline{v} = 0\}$ so it may be identified with the space of oriented lines in $\mathbb{R}^3$. Also $T\mathbb{P}^1$ fibres over $\mathbb{P}^1$ and we may take coordinates (η, ζ) on $T\mathbb{P}^1$ where ζ is a coordinate on $\mathbb{P}^1$ and η is a fibre coordinate. There is a real structure on $T\mathbb{P}^1$; in terms of the above coordinates it is $\tau:(\eta,\zeta) \mapsto (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$, but it is easier to think of it as just reversing the orientation of oriented lines in $\mathbb{R}^3$. We define line bundles L^t of degree 0 over $T\mathbb{P}^1$ by letting L^t be the bundle with transition function $\exp(t\eta/\zeta)$.

For each SU(2)-monopole there is just one associated spectral curve S in $T\mathbb{P}^1$. It satisfies:-

(i) S is compact and has equation $\eta^k + a_1(\zeta)\eta^{k-1} + \ldots + a_k(\zeta) = 0$ where each a_i is a polynomial of degree $2i$

(ii) L^2 is trivial over S; or equivalently (since $\deg L^2 = 0$), $H^0(S, L^2) \neq 0$

(iii) S is preserved by the real structure τ

(iv) S has no multiple components

(v) (nondegeneracy condition) $H^0(S, L^t(k-2)) = 0$ for $0 < t < 2$

A parameter count gives the dimension of the moduli space of charge k SU(2)-monopoles as 4k-1.

These results have been extended to the case of SU(n)-monopoles with symmetry broken to U(1)x...x U(1) by Michael Murray [3] who showed that such monopoles were generically determined by n-1 spectral curves (satisfying certain constraints) in minitwistor space.

We may also consider monopoles with nonmaximal symmetry breaking i.e. symmetry broken to a nonabelian subgroup. In particular consider SU(3) monopoles with symmetry to U(2). As SU(3) is the QCD gauge group such monopoles may be of particular physical interest.

We now have only one spectral curve (as opposed to two curves for U(1) x U(1) symmetry breaking). This curve satisfies conditions (i) and (iii) above; however the condition that there is a nontrivial element of $H^0(S,L^2)$ is replaced by the requirement that $H^0(S,L^3(1,1)) \neq 0$ (for the charge 21 monopole). Parameter counting, using results from algebraic geometry about the dimension of linear systems on algebraic curves, suggests that the charge 21 moduli space should have dimension $\leq 121-1$; in fact the charge 2 moduli space should have dimension precisely 11 (or 8 once we fix the centre of the monopole in $\mathbb{R}^3$). This agrees with a result of E. Weinberg [5] (Weinberg includes an S^1 phase to get 12 parameters).

Further investigations concerning nondegeneracy conditions suggest that the 7-dimensional space of SU(2) charge 2 monopoles should arise as a boundary of the 8-dimensional space of SU(3) minimal symmetry breaking charge 2 monopoles. Now it is known [2] that SU(2) charge k monopoles are equivalent to triples.

(T_1, T_2, T_3) of k x k-matrix valued functions on [0,2] satisfying:-

(1) $T_i^*(t) = -T_i(t)$

(2) $T_i(2-t) = -T_i(t)$

(3) T_i is analytic on (0,2) with simple poles at $t = 0,2$

(4) $\frac{dT_1}{dt} = [T_2, T_3]$ and cyclically (Nahm's Equations).

(5) The residues of the T_i at $t = 0,2$ give an irreducible representation of SU(2).

The pole at t=2 corresponds to the bundle L^2 being trivial over the spectral curve S. Condition (2) reflects the quaternionic nature of SU(2) ($\cong$ Sp(1)). In the SU(3) case, therefore, we should drop these conditions. The resulting modified system of Nahm's equations may be solved (in the charge 2 case) explicitly using SO(3) and SU(2) symmetries and Jacobi elliptic functions. The moduli space of centred SU(3) charge 2 monopoles is then an 8-dimensional space with the SU(2) moduli space as a boundary. The moduli space includes a spherically symmetric monopole and a 3-parameter family of axisymmetric monopoles; this agrees with results of Ward arrived at via twistor theory [4]; (Ward considers uncentred monopoles and so gets a 6-parameter family of axisymmetric solutions).

A.S. Dancer

References

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