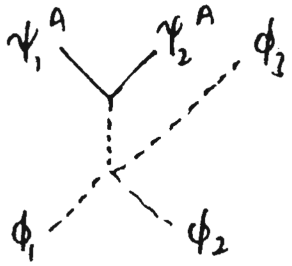
Twistor translation of Feynman vertices

Progress has been made in the programme of translating general Feynman diagrams into twistor diagrams. The advance comes through observing the *breaking of conformal invariance* in elementary Feynman amplitude calculations (that is, conformal symmetry-breaking even before renormalisation considerations come in). An example shows this explicitly:

Consider the Feynman diagram



where dashed lines indicate scalar fields and solid lines spin-1/2 fields. Such a diagram arises in the standard model from the presence of terms in the interaction Lagrangian of form

$$\psi_A \bar{\psi}^A \phi, \qquad (\phi\bar{\phi})^2$$

The Feynman diagram as drawn indicates the integral

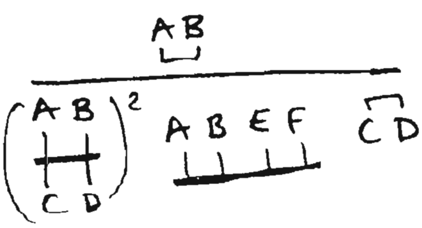
$$\int d^4x\, d^4y\; \psi_1^{\ A}(x)\psi_{2A}(x)\, \Delta_F(x-y)\, \phi_1(y)\phi_2(y)\phi_3(y)$$

considered as a functional of the external fields, where ϕ_1, ϕ_2 are of positive frequency and all the others of negative frequency.

We now choose particular fields, specified by corresponding twistor 1-functions:

$$\psi_1^{\ A}(x) \leftrightarrow \frac{1}{\substack{W\\|\\C}\left(\substack{W\\|\\D}\right)^2}, \quad \psi_2^{\ A}(x) \leftrightarrow \frac{1}{\left(\substack{W\\|\\C}\right)^2\substack{W\\|\\D}}, \quad \phi_1 = \phi_2 \leftrightarrow \frac{1}{\substack{A\\|\\2}\;\substack{B\\|\\2}}, \quad \phi_3 \leftrightarrow \frac{1}{\substack{E\\|\\2}\;\substack{F\\|\\2}}$$

Space-time calculation then yields the result of the Feynman integral, namely



This is $I^{\alpha\beta}$-dependent although scale-invariant (of homogeneity 0 in $I^{\alpha\beta}$.)

Hence we know that any proposed twistor diagram for this amplitude must involve $I^{\alpha\beta}$ in some way. Twistor diagrams therefore cannot hope to give a manifestly conformally invariant description of amplitudes in general: they can however make the dependence on $I^{\alpha\beta}$ explicit.

We can now arrive at the same conclusion by a different and more general argument. Note that the relation

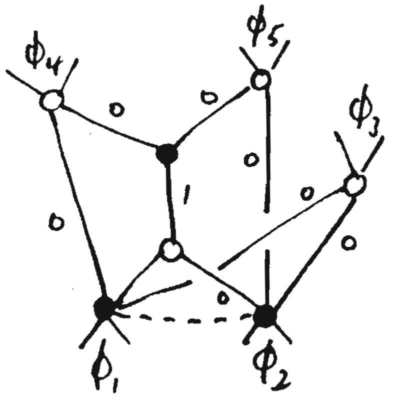
$$\iint d^4x\, d^4y\, \left(\Box_x \psi_1^{A}(x)\,\psi_{2A}(x)\right)\Delta_F(x-y)\,\phi_1(y)\,\phi_2(y)\,\phi_3(y)$$

$$= \int d^4x\, \psi_1^{A}(x)\,\psi_{2A}(x)\,\phi_1(x)\,\phi_2(x)\,\phi_3(x)$$

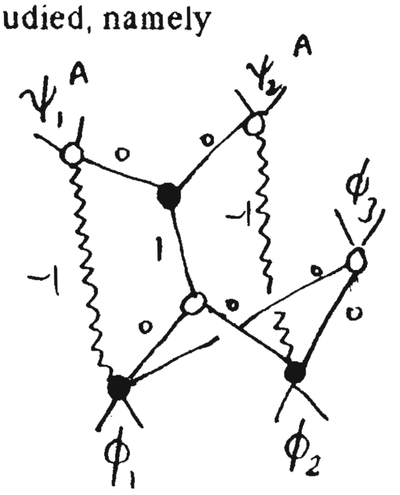
means that any proposed twistor representation of the Feynman integral above must be related via a differential operator with a representation of

the integral $\int d^4x\ \phi_4(x)\,\phi_5(x)\ \phi_1(x)\,\phi_2(x)\,\phi_3(x)$

Now there are many possible representations of this "ϕ^5" integral. One of them is



and this one allows the inverse of this differential operator to be applied, at least formally. This yields a candidate twistor diagram for the Feynman integral being studied, namely



But is there a contour for this diagram yielding the Feynman amplitude? NO. Proof by contradiction: suppose there were, then we could use it to effect an integration by parts of the "ϕ^5" diagram. But now take the limit as $\phi_5(x)$ moves towards the constant field, i.e the elementary state based at $I^{\alpha\beta}$. If the integration by parts were valid, this limit would be zero identically. But this limit must in fact be the integral

$$\int d^4x\ \phi_1(x)\,\phi_2(x)\,\phi_3(x)\,\phi_4(x)$$

which is non-zero in general.

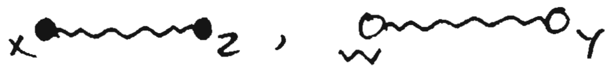
Clearly this argument is not specific to this diagram and can be applied in general in situations where there are more than two in- or out-fields.

But suppose we allow ourselves to add further elements to the integral which involve $I^{\alpha\beta}$ explicitly in the singularity structure. Then the argument above can no longer be used to yield a contradiction. The limiting case of the constant field cannot taken since the contour may pinch in this limit.

In fact there is a natural candidate for the form to be taken by these new elements, namely *boundaries at infinity*, i.e. boundaries on subspaces

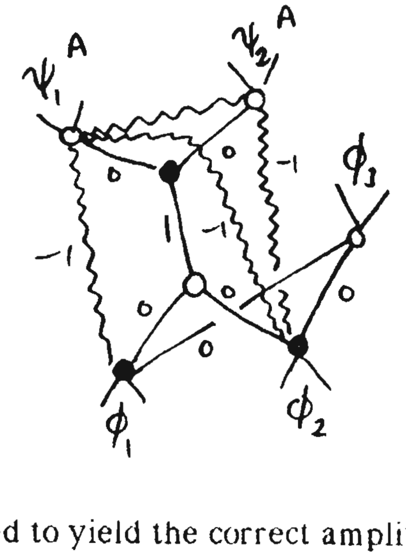
$$\{\overline{XZ} = 0\} \quad \text{or} \quad \{\underline{WY} = 0\}$$

represented in diagrams by

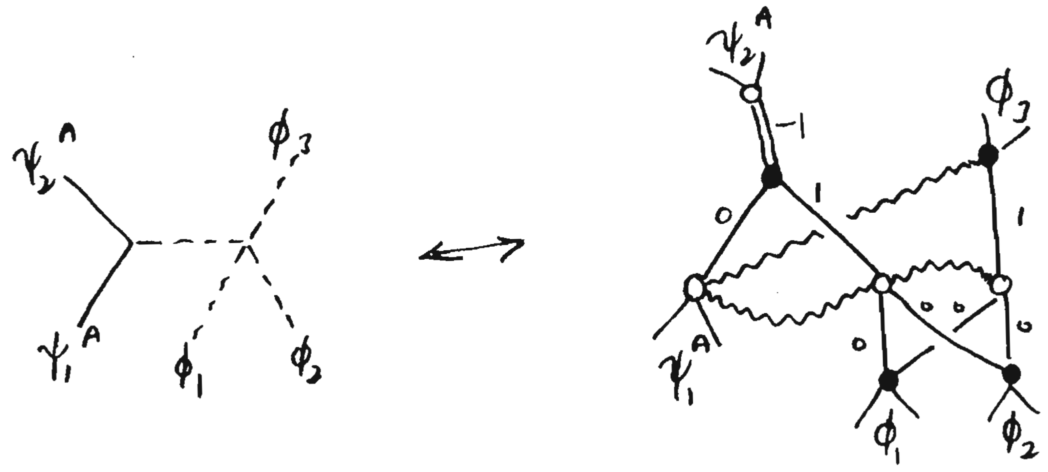


Note that such boundaries are "invisible to" the ∇, $\Box$ operators, which is why it is permissible to add them without upsetting the essential differential equation satisfied by (2).

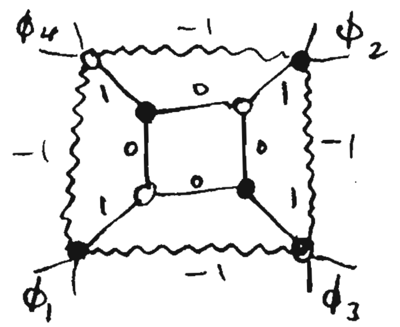
In fact one finds explicitly that



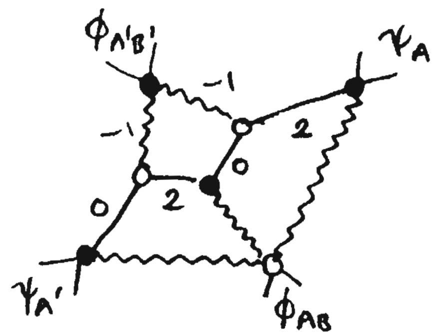
can be integrated to yield the correct amplitude. Similarly we may consider the channel in which the spin-1/2 fields have are of opposite frequency types and find a correspondence



We are now faced with the problem, familiar from earlier work, of finding a twistor interpretation of the crossing relations. In **TN 28**, analysis of the double-box led to the observation that the diagram



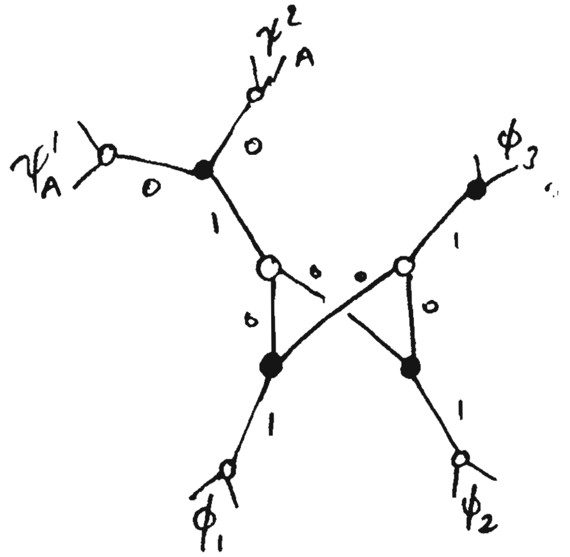
contained all channels for first-order ϕ^4 scattering within it; similarly for the diagram



with respect to Coulomb scattering.

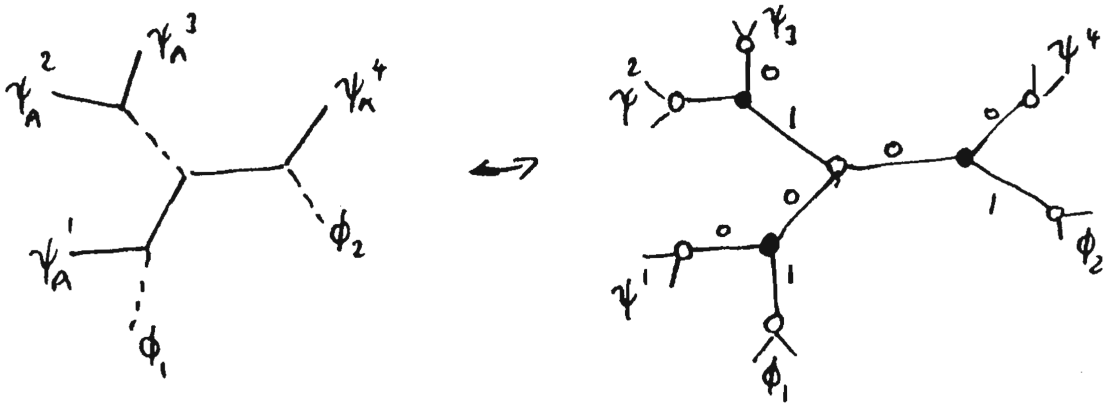
We might rephrase this observation in terms of asserting the existence of a *skeleton diagram* formed solely out of pole singularities, defining a twistor differential form which when combined with an appropriate collection of boundary prescriptions can yield the amplitude in any of the crossing-related channels.

Skeleton diagrams can be given for all the $(2 \to 2)$ processes hitherto studied. We now notice that the diagrams (3) and (4) above can be regarded in this sense as different realizations of the skeleton diagram



It has not yet been checked in detail that boundary-contours exist to yield *all* channels, but this seems very probable.

Attempted generalizations are now readily suggested. In particular, it appears that the process involving three "Yukawa interaction" vertices has a correspondence with a skeleton diagram given by



(This is certainly valid for *some* channels).
The significance of this process lies in regarding the external fields as test functions for the product of three Feynman propagators. Thus, looking at this diagram in all possible channels is equivalent to giving the full information of a general Feynman diagram vertex. Composition of these vertices into general Feynman diagrams should then be possible.

At present there is no indication of how the boundaries corresponding to each channel may be defined (they are certainly not uniquely defined.) Thus we are still faced with the problems of definition that have always arisen in diagram theory. However, this skeleton diagram concept at last suggests a framework which can include everything known so far and has the potential for systematic generalization.

Clearly we shall in general have to modify the definitions to introduce *inhomogeneous* poles and boundaries - possibly also logarithmic factors rather than boundaries - if we are to incorporate the inhomogeneity which eliminates infra-red divergences at first-order level. Note that to make the *boundaries at infinity* inhomogeneous would imply introducing a *mass* parameter.

There are then at least three directions which the existing results suggest for investigation:

(1) establishing the general vertex for Yukawa and ϕ^4 interaction, and then studying **Feynman loop diagrams** within these theories. Can the introduction of inhomogeneity eliminate ultra-violet divergences systematically?

(2) generalising these higher order calculations to **massless electroweak** theory. As may be seen from the Coulomb "skeleton", the feature that seems to be emerging is that gauge fields appear in the boundary specifications and not in the skeleton. This suggests the hopeful picture of a twistor calculus in which gauge fields are all absorbed into geometry in a manifestly gauge invariant way. In particular, for *pure* gauge field scattering the skeleton should virtually disappear, leaving only the specification of a bounded region of integration. This might give the closest link between diagram theory and the conformal field theory picture.

(3) The primary significance of the Yukawa interaction in the standard model is that it provides the mechanism for **massive fields** to arise. The essential idea is that the scalar field is, in the zeroth order, the constant field. The contours for the twistor diagrams we have written down will in general "pinch" as the external field tends towards the constant field (i.e. the elementary state based at $I^{\alpha\beta}$) but by studying this limit it may be possible to find a modification of twistor geometry at $I^{\alpha\beta}$ which allows the massive fields to emerge as a finite limit. The massive fields have already been described by twistor integrals involving inhomogeneous *poles* at infinity and it should be possible to relate these to the suggested formalism.

The combination of all three directions of generalization would amount to a general theory for translating Feynman diagrams into twistor diagrams.

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