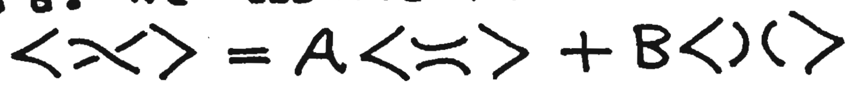
Spin Networks and the Jones Polynomial

by Louis H. Kauffman

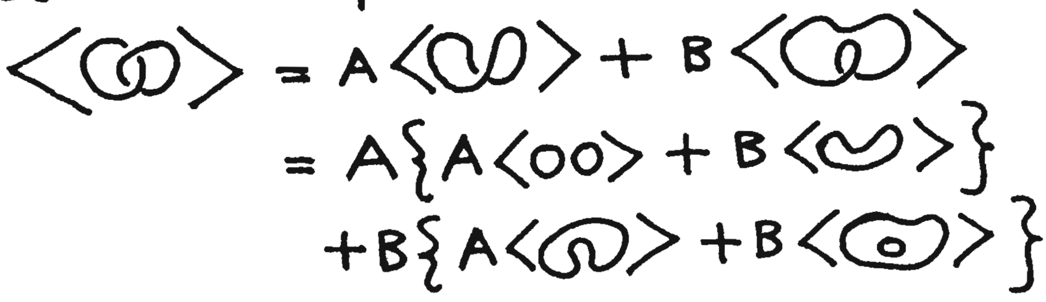
The purpose of this note is to point out that the binor calculus of Roger Penrose is a special case of the Jones polynomial, and that this generalization naturally involves the movement from SL(2) invariant tensor diagrams to SL(2)q invariant tensor diagrams, where SL(2)q denotes the quantum group!

In order to see this, I first recall the bracket states model for the Jones polynomial [3], [4], [2]. We are given a function $\langle K \rangle$ defined on un-oriented link diagrams such that $\langle K \rangle$ is a polynomial in A, B and d. We assume that

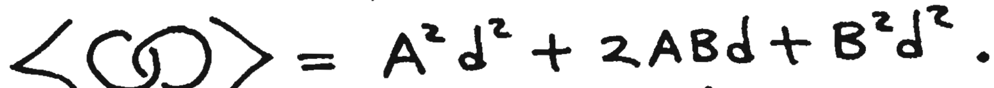


and

The small diagrams are parts of larger diagrams, identical except as indicated. For example,

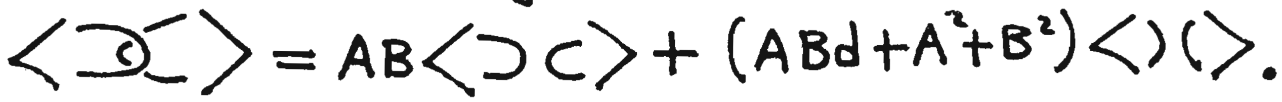


$$= A^2d^2 + ABd + BAd + B^2d^2$$



(A, B and d commute.)

To create a topological invariant, we use the following formula:

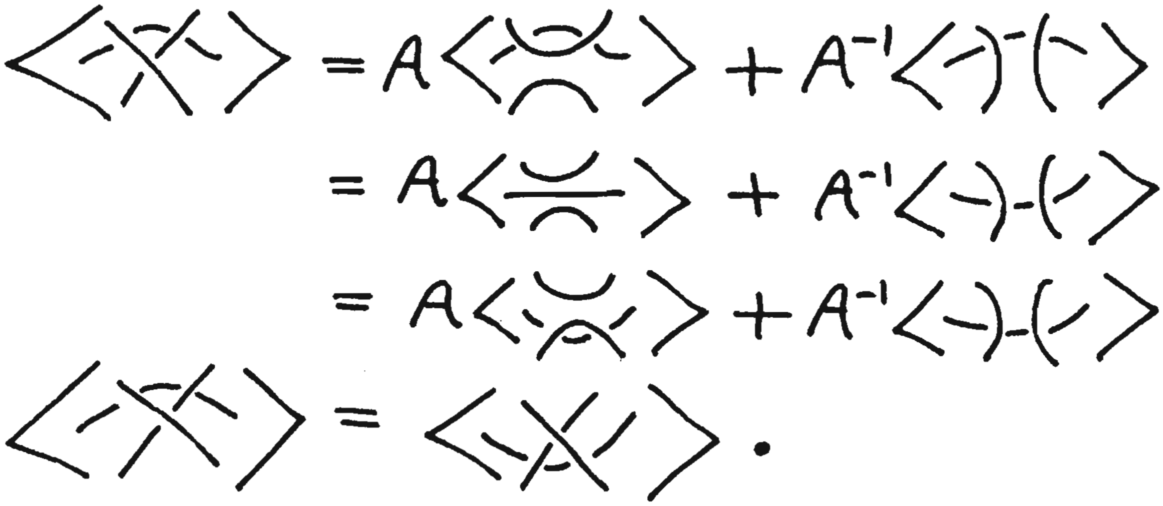


(easily proved)

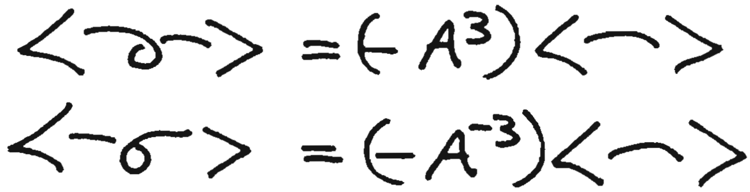
Thus *if* $B = A^{-1}$ and $d = -A^2 - A^{-2}$

then

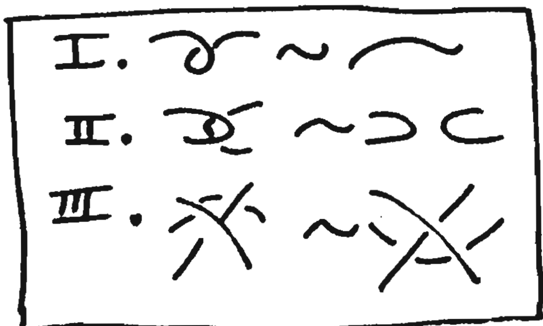
It then follows directly that



Finally,



and so $\langle K \rangle$ is an invariant of Reidemeister moves II and III, and it is well-behaved on I.



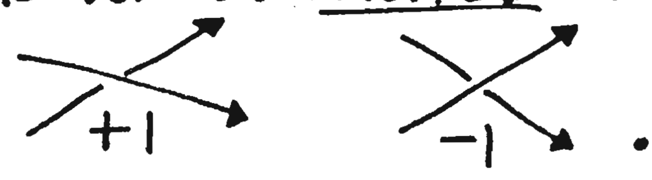
These moves generate topological equivalence of knots and links.

REIDEMEISTER MOVES

By suitable normalization, one gets the [2] Jones polynomial: $V_K(t) = f_K(t^{-1/4})$

where $f_K(A) = (-A^3)^{-w(K)} \langle K \rangle / \langle O \rangle$

and $w(K)$ is the sum of the crossing signs for an *oriented* link K:



Now we shall stay with the bracket, and observe that if $A=-1$, then

1. $\langle \times \rangle = -\langle \asymp \rangle - \langle)(\rangle = \langle \times \rangle$
2. $\langle O \rangle = -2$.

Thus <u>bracket at $A=-1$ reproduces the binor formalism</u>: $\boxed{\begin{matrix} \times + \asymp +)(= 0 \\ O = -2 \end{matrix}}$

Now the binors were created to be topologically well-defined $SL(2)$ invariant tensor diagrams. In particular, we can let $\underset{a\,b}{\sqcap} = \varepsilon_{ab}$, $\overset{a\,b}{\sqcup} = \varepsilon^{ab}$ denote spinor epsilons so that $\varepsilon_{12}=1$, $\varepsilon_{21}=-1$, $\varepsilon_{11}=\varepsilon_{22}=0$ (two indices) and $\varepsilon^{ij}=\varepsilon_{ij}$.

Then let $\cap = \sqrt{-1}\,\sqcap$, $\cup = \sqrt{-1}\,\sqcup$ and the Fierz identity $\asymp =)(- \times$ becomes $\asymp +)(+ \times = 0$ [under the convention that the crossing contributes a (-1) sign so that $\overset{a\quad b}{\underset{c\quad d}{\times}} = -\delta^a_d\,\delta^b_c$]. And $O = O = (\sqrt{-1})^2\, \sqcap\sqcup = -2$.

In matrix terms, $\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $\sqrt{-1}\,\varepsilon = \begin{bmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}$. <u>The loop value O is the sum of the squares of the entries of this matrix.</u>

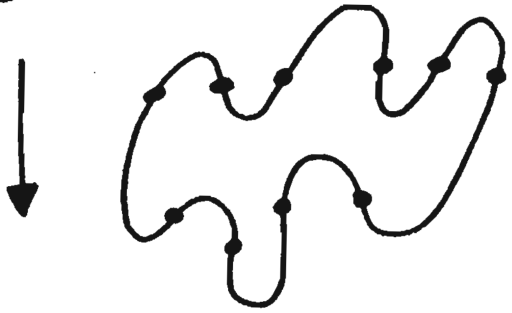
In order to create a similar model for the bracket, we need to deform $\sqrt{-1}\,\varepsilon$ so that the sum of the squares of its entries is $d = -A^2 - A^{-2}$. Therefore, let $\tilde{\varepsilon} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix}$ and let $M = \begin{bmatrix} 0 & \sqrt{-1}\,A \\ -\sqrt{-1}\,A^{-1} & 0 \end{bmatrix}$.

Let $= M_{ab}$ and $= M^{ab} = M_{ab}$.

Then, we have $M_{\cdot\cdot}M^{\cdot\cdot} =]]$, i.e. $M^2 = I$, and

$$= M_{ab}M^{bc} = \delta_a^c = .$$

Thus any version of the loop will give $d = -A^2 - A^{-2}$



,

and one can think of $\langle K \rangle$ as a vacuum-vacuum expectation [7] with creations, annihilations and interactions. Bracket tells the exact form of the braiding matrices:

$$R^{ab}_{cd} = \; = A \; + A^{-1}$$

and these can be seen directly to satisfy the Yang-Baxter equation [1] (corresponding to the III-move.)

The Quantum Group

$$SL(2) = \{\mathcal{U} \mid \mathcal{U}\varepsilon\mathcal{U}^T = \varepsilon\}.$$

Therefore consider $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with associative, possibly non-commutative entries, and ask for

$$\boxed{\begin{array}{l} P\,\hat{\tilde{\varepsilon}}\,P^T = \hat{\tilde{\varepsilon}} \\ P^T\,\hat{\tilde{\varepsilon}}\,P = \hat{\tilde{\varepsilon}} \end{array}}.$$

Check that this is equivalent to the equations:

$$\boxed{\begin{array}{ll} ca = q\,ac & db = q\,bd \\ ba = q\,ab & dc = q\,cd \\ bc = cb & \\ ad - da = (q^{-1} - q)\,bc & \\ ad - q^{-1}bc = 1 & \end{array}} \quad q = \sqrt{A}.$$

This algebra is the dual $\mathcal{U}^*$ of the quantum universal enveloping algebra $\mathcal{U}_q sl_2$ [1]. Thus the "~~q~~quantum group" $\mathcal{U}^* = SL(2)_q$ appears naturally as the algebra of symmetries of abstract topological tensor diagrams generalizing the binor calculus.

This heralds a corresponding generalization of spin networks, and perhaps the Spin Geometry Theorem [5], [6].

References

1. Drinfeld, V. G. "Quantum Groups", Proc. Int. Cong. Math., Berkeley, Calif. USA, pp. 798–820 (1986).

2. Jones, V.F.R. "A polynomial invariant for links via von Neumann algebras". Bull. Amer. Math. Soc. 12 pp. 103-112 (1985).

3. Kauffman, L.H. "State models and the Jones polynomial". Topology 26 pp. 395-407 (1987).

4. Kauffman, L.H. "Statistical mechanics and the Jones polynomial", Amer. Math. Soc. Cont. Math. Series #78 (1988), pp. 263-297.

5. Moussouris, J.P. "Quantum models of space-time based on recoupling theory", Thesis - Oxford Univ. (1983).

6. Penrose, R. "Angular momentum - an approach to combinatorial space-time", from Quantum Theory and Beyond ed. by T. Bastin. Camb. Univ. Press. (1971).

7. Witten, E. "Quantum field theory and the Jones polynomial", Commun. Math. Phys. 121, 351-399 (1989).

Department of Mathematics, Statistics and Computer Science
The University of Illinois at Chicago
Chicago, Illinois 60680