Twistor Newsletter

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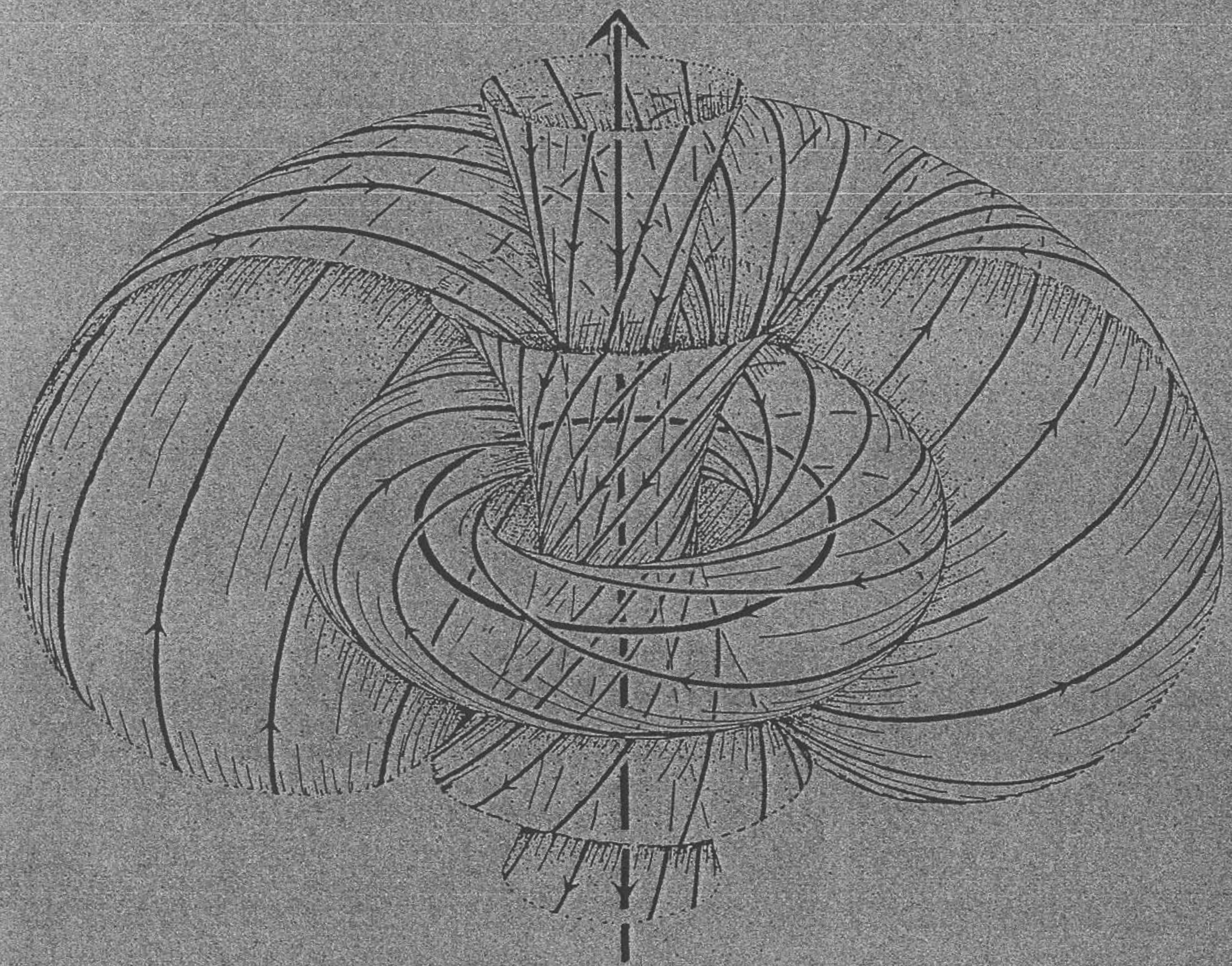

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A Twistorial Approach to the Full Vacuum Equations

In an attempt to explore possible connections between Ashtekar variables and hypersurface twistors, we have been led to a (tentative) new approach to the twistorial description of general relativity.

Let $\mathcal{M}$ be a complexified (analytic) space-time, and $\mathcal{H}$, a complexified (analytic) spacelike hypersurface in $\mathcal{M}$. A point in the projective hypersurface twistor space $\mathbb{P}\mathcal{T} = \mathbb{P}\mathcal{T}(\mathcal{H})$ is represented as an α-curve in $\mathcal{H}$. An α-curve is a (complex) null curve in $\mathcal{H}$ with tangent vector of the form $\tilde{\pi}^A \pi^{A'}$, where

$$\tilde{\pi}_A = t_A{}^{A'} \pi_{A'} ,$$

the vector t^a being normal to $\mathcal{H}$ (and can be normalized, if so desired, by

$$t_a t^a = 2, \qquad \text{so that } t_A{}^{B'} t_{B'}{}^C = -\varepsilon_A{}^C \ \& \ t_{A'}{}^B t_B{}^{C'} = \varepsilon_{A'}{}^{C'}$$

whence $\tilde{\tilde{\pi}}_{A'} = -\pi_{A'}$, with $\tilde{\lambda}_{A'} = t_{A'}{}^A \lambda_A$ etc.), and where we require

$$\tilde{\pi}^A \pi^{A'} \nabla_{AA'} \pi_{B'} = 0 . \qquad \text{..... (A)}$$

The non-projective hypersurface twistor space $\mathcal{T} = \mathcal{T}(\mathcal{H})$ is a line bundle over $\mathbb{P}\mathcal{T}$, with fibres given by the scalings for $\pi_{A'}$. (See Spinors & Space-Time, vol. 2 pp. 215–218, R.P. & W.R.)

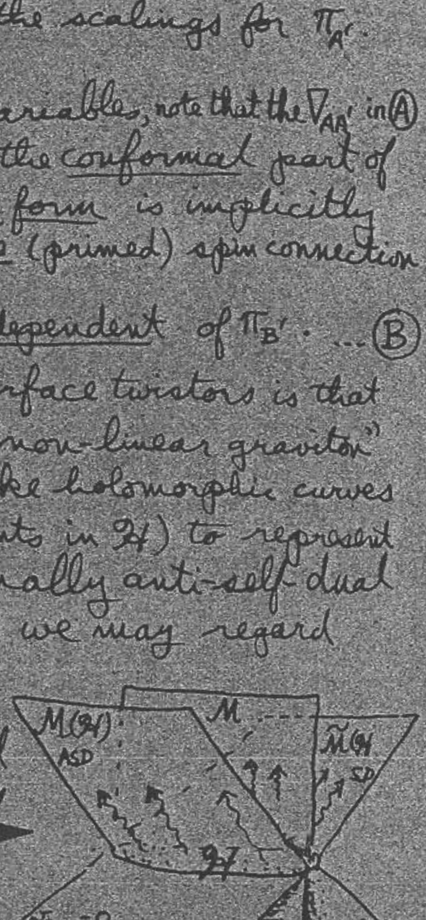
As a point of contact with Ashtekar's variables, note that the $\nabla_{AA'}$ in (A) is his (primed) spin connection — though only the conformal part of it is being used here — and his soldering form is implicitly involved in $\tilde{\pi}^A \pi^{A'} \nabla_{AA'}$. To obtain the entire (primed) spin connection we should need to incorporate

$$\tilde{\pi}^A \pi^{A'} \nabla_{AA'} \nu_{B'} \qquad \text{with } \nu_{B'} \text{ independent of } \pi_{B'} . \ \text{... (B)}$$

A helpful concept, in relation to hypersurface twistors is that of "Heaven on Earth" $\mathcal{M}(\mathcal{H})$, which is the "non-linear graviton" construction applied to $\mathbb{P}\mathcal{T}$ (or $\mathcal{T}$); i.e. we take holomorphic curves in $\mathbb{P}\mathcal{T}$ (homologous to those representing points in $\mathcal{H}$) to represent points in $\mathcal{M}(\mathcal{H})$. Note that $\mathcal{M}(\mathcal{H})$ is conformally anti-self-dual (but not conformal to vacuum, in general), and we may regard

$$\mathcal{H} \subset \mathcal{M}(\mathcal{H}) .$$

(We could define $\tilde{\mathcal{M}}(\mathcal{H})$ similarly, as a self-dual space, where primed and unprimed spinor indices are interchanged.) The picture may be helpful. If $\mathcal{M}$ is vacuum, we can regard $\mathcal{M}$ as the Einstein evolution of $\mathcal{H}$, whereas $\mathcal{M}(\mathcal{H})$ is a conformal manifold evolved from the (conformally invariant part of) the same data at $\mathcal{H}$, but evolved according to $\tilde{\Psi}_{A'B'C'D'} = 0$. The space $\mathcal{M}(\mathcal{H})$ is "equivalent information" to $\mathcal{T}$, and



it provides a useful way of studying the global structure of $\mathcal{T}$. (Similarly, the conformally self-dual space $\tilde{\mathcal{M}}(\mathcal{H})$, evolved from $\mathcal{H}$ according to $\Psi_{ABCD}=0$, gives equivalent information to the "dual" hypersurface twistor space defined by β-curves (compl. conj. of Ⓐ) on $\mathcal{H}$.)

The complex conformal space $\mathcal{M}(\mathcal{H})$ provides a good part of the initial data needed at $\mathcal{H}$ for the evolution of Einstein's equations (to give $\mathcal{M}$). However, two additional ingredients are needed:

① a conformal scale (equivalent to Ⓑ above)

② the location of $\mathcal{H}$ itself, within $\mathcal{M}(\mathcal{H})$.

Here we make use of an idea (cf. L.J.M. in TN 28: "Hypersurface Twistors"; C.R.LeB. TN 12 "Metrics as cohomology classes") whereby ① and ② are coded as two solutions of the conformally invariant wave equation on $\mathcal{M}(\mathcal{H})$

$$(\Box + \tfrac{R}{6})\Omega = 0, \qquad (\Box + \tfrac{R}{6})\chi = 0.$$

Using $g_{ab} \mapsto \Omega^2 g_{ab}$, we scale $\Omega \mapsto \Omega^{-1}\Omega = 1$ (so that R is scaled to zero) and $\chi \mapsto \Omega^{-1}\chi$, and we use:

$$\chi = 0 \quad \text{to define the location of } \mathcal{H}.$$

The freedom in Ω is precisely the freedom in choosing a conformal scale on $\mathcal{M}(\mathcal{H})$ so that $R=0$; the freedom in χ is the freedom in the choice of $\mathcal{H}$ within $\mathcal{M}(\mathcal{H})$ together with the choice of "lapse function", provided by the choice of normal (where now t_a need not be normalized by $t_a t^a = 2$)

$$t_a = \nabla_a \chi \quad \text{at } \mathcal{H}$$

in $\mathcal{M}(\mathcal{H})$ — or, equivalently, in $\mathcal{M}$. (Note that $\chi=0$ at $\mathcal{H}$, together with $\nabla_a\chi$, give initial data for $(\Box+\frac{R}{6})\chi=0$ in $\mathcal{M}(\mathcal{H})$.)

We could specify Ω and χ as two elements of $H^1(\mathcal{O}(-2))$ on $\mathbb{P}\mathcal{T}$ — but we prefer to use a <u>bundle</u> $\mathbb{B}$ over $\mathbb{P}\mathcal{T}$ (or $\mathcal{T}$). We take $\mathbb{B}$ as an affine rank 2 bundle whose fibres are solutions of

$$\nabla_A \nu_{A'} = \nabla_{AA'}\chi \quad \text{on an } \alpha\text{-plane in } \mathcal{M}(\mathcal{H}) \ldots Ⓒ$$

(Note that, being ASD, $\mathcal{M}(\mathcal{H})$ has α-planes — which meet $\mathcal{H}$ in the α-curves — and each α-plane corresponds to a point of $\mathbb{P}\mathcal{T}$.) Here we define

$$\nabla_A = \pi^{A'} \underbrace{\nabla_{AA'}}_{\text{derivative operator in } \mathcal{M}(\mathcal{H})},$$

which differentiates within the α-plane, and we assign a homogeneity -1 to $\nu_{A'}$, with respect to $\pi_{A'}$. (Thus, under $\pi_{A'} \mapsto \zeta\pi_{A'}$,

we take $\nu_{A'} \mapsto \varsigma^{-1}\nu_{A'}$.) Note that χ has homogeneity 0.
Observe that since $\tilde{\Psi}_{A'B'C'D'} = 0$ in $\mathcal{M}(\mathcal{H})$, we have

$$\nabla_A \nabla_B \nu_{C'} - \nabla_B \nabla_A \nu_{C'} = 0$$

which demands

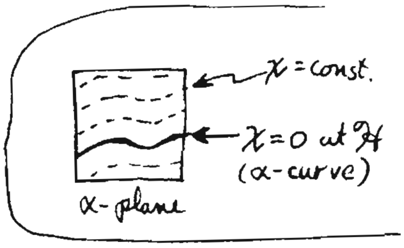
$$\Box \chi = 0$$

and which we have taken to be so. We note also that

$$X[\nu] = \chi - \nu_{A'}\pi^{A'} \in \mathbb{C}$$

is constant on the α-plane, so we have a bundle map

$$X : \mathcal{B} \longrightarrow \mathcal{T} \times \mathbb{C}.$$



This bundle map leads to the exact sequence of affine bundles

$$0 \longrightarrow \mathcal{B}_0 \longrightarrow \mathcal{B} \xrightarrow{X} \mathcal{O} \longrightarrow 0. \qquad \ldots \text{(D)}$$

The affine-bundle $\mathcal{B}_0$ is the zero set of X and encodes the cohomology class of χ alone, and the vector bundle $\hat{\mathcal{B}}$, the underlying translation bundle, or bundle of displacements of $\mathcal{B}$, encodes Ω alone.

The line bundle $\mathcal{B}_0$ has $\mathcal{O}(-2)$ as its bundle of displacements, so that (D) leads to an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \hat{\mathcal{B}} \longrightarrow \mathcal{O} \longrightarrow 0, \qquad \ldots \text{(E)}$$

where points of the fibre of $\hat{\mathcal{B}}$ are differences between solutions of (C), i.e. solutions of $\nabla_A \nu_{A'} = 0$ on $\mathcal{M}(\mathcal{H})$. This codes the needed information of the primed spin connection. The cohomology class of $\Omega \in H^1(\mathcal{O}(-2))$ is the extension class that determines the sequence (E).

The cohomology class of $\chi \in H^1(\mathcal{O}(-2))$ determines $\mathcal{B}_0$ (see M.G.E. in TN 14 "The Penrose Transform without Cohomology"). Geometrically $\mathcal{B}_0$ encodes the location of $\mathcal{H}$ in $\mathcal{M}(\mathcal{H})$ as follows. If $\mathcal{L}$ is a holomorphic curve in $\mathbb{P}\mathcal{T}$ corresponding to a point L in $\mathcal{M}(\mathcal{H})$, a section $\mathcal{S}$ of $\mathcal{B}$ over $\mathcal{L}$ is a $\nu_{A'}(\pi_{B'})$ at L with homogeneity -1 in $\pi_{B'}$. The spinor $\nu_{A'}$ must therefore vanish and so X will map all of $\mathcal{S}$ to the value of χ in $\mathbb{C}$, which is 0 iff $L \in \mathcal{H}$. The ~~total~~ space of $\mathcal{B}$ can be canonically identified with the spin bundle of $\mathcal{M}(\mathcal{H})$ and $\mathcal{B}_0$ with its restriction to $\mathcal{H}$.

How does Einstein's equation fit into all this? How do we formulate the constraint equations and evolution equations? The following curious fact may be useful. Let $(\rho^A, \sigma_{A'})$ be a <u>local</u> <u>twistor</u> which is <u>constant</u> on the α-plane (consistent because $\tilde{\Psi}_{A'B'C'D'} = 0$). We have

$$\nabla_A \rho^B = -i\pi^{A'}\sigma_{A'}\,\varepsilon_A{}^B$$
$$\nabla_A \sigma_{B'} = -i P_{ABA'B'}\pi^{A'}\rho^B$$
on an α-plane in $\mathcal{M}(\mathcal{H})$
(N.B: $P_{ABA'B'} = \Phi_{ABA'B'}$ on $\mathcal{M}(\mathcal{H})$)
$= P_{(AB)(A'B')}$

(These equations of local twistor transport are the Ward transform of $T^*(-1)\mathcal{T}$, see C.R.LeB. in Class. & Quant. Grav. 1986 "Thickenings and Gauge fields, and N.M.J.W. in " " " " 1985 "Real methods in twistor theory.) Consider $\tilde{\sigma}_A = t_A{}^{A'}\sigma_{A'}$ (with t_a not necessarily normalized); then we have

$$\nabla^A \tilde{\sigma}_A = 0 \quad \text{at } \chi = 0 \text{ on } \mathcal{M}(\mathcal{H}) \qquad (F)$$

by virtue of $\Box\chi = 0$ and of

$$P_{ABA'B'}\,t^{AA'} = 0 \quad \text{at } \chi=0 \text{ on } \mathcal{M}(\mathcal{H}). \qquad (G)$$

Equation (G) states the <u>constraint equations</u> for Einstein's theory, the $P_{ABA'B'}$ for $\mathcal{M}(\mathcal{H})$, at $\mathcal{H}$, being a linear combination of

$$\underset{\bullet}{\Psi}_{ABCD}\,t^C{}_{A'}\,t^D{}_{B'}\,,\quad \underset{\bullet}{\tilde{\Psi}}_{A'B'C'D'}\,t^{C'}{}_A\,t^{D'}{}_B\,,\quad \underset{\bullet}{\Phi}_{ABA'B'}\,,\quad \underset{\bullet}{\Lambda}\,t_{AA'}t_{BB'}$$

where the curvature quantities with dots (•) beneath them denote the <u>physical</u> quantities for $\mathcal{M}$. (Before, we had been using the quantities for $\mathcal{M}(\mathcal{H})$.) The terms in $\underset{\bullet}{\Psi}_{ABCD}$ and $\underset{\bullet}{\tilde{\Psi}}_{A'B'C'D'}$ disappear, in $P_{ABA'B'}\,t^{AA'}$, and we are left only with the Ricci tensor parts (because of the symmetry of $\underset{\bullet}{\Psi}_{ABCD}$, $\underset{\bullet}{\tilde{\Psi}}_{A'B'C'D'}$). Thus, by (F),

$$\tilde{\sigma}_A = \nabla_A \xi \quad \text{on } \chi = 0 \text{ (in } \mathcal{M}(\mathcal{H})\text{), for some } \xi$$

states the constraint equations for the Einstein theory!

What is ξ? We can find a "potential" for $P_{ABA'B'}$ at $\chi=0$ (on an α-plane in $\mathcal{M}(\mathcal{H})$) by solving

$$\{\nabla_A\nabla_B + \pi^{A'}\pi^{B'}P_{ABA'B'}\}\,\eta = P_{ABA'B'}\,\nu^{A'}\pi^{B'}$$

and then we find that we can put

$$\xi = \sigma_{A'}(\nu^{A'} - \eta\pi^{A'}) + \rho^A(i\nabla_A\eta) = \sigma_{A'}\mu^{A'} + \rho^A\lambda_A$$
$$= S^\alpha W_\alpha \qquad \text{where } \begin{cases} S^\alpha = (\rho^A, \sigma_{A'}) \\ W_\alpha = (i\nabla_A\eta,\ \nu^{A'} - \eta\pi^{A'}) = (\lambda_A, \mu^{A'}) \end{cases}$$

on the α-plane in $\mathcal{M}(\mathcal{H})$.

Here W_α is not local twistor constant on the α-plane because of

$$\nabla_A \mu^{B'} = i\lambda_A \pi^{B'} + \nabla_A{}^{B'}\chi$$
$$\nabla_A \lambda_B = i P_{ABA'B'}\mu^{A'}\pi^{B'}$$

By Einstein (constraint equs.): integrable to 1st order away from $\chi = 0$.

This suggests that we do the following: instead of using $\mathbb{B}$, why not use $\mathcal{A}$, where the fibre over a pt. Z^α of $\mathcal{T}$ is a $W_\alpha = (\lambda_A, \mu^{A'})$ satisfying these two equations (or rather, these equs. contracted with π^A). This is an "affinized" version of the cotangent bundle $T^*(-1)\mathcal{T}$, of $\mathcal{T}$, referred to above, and we can construct the bundle of displacements $\hat{\mathcal{A}}$ of $\mathcal{A}$ (as with $\hat{\mathbb{B}}$ above) so that $\hat{\mathcal{A}} = T^*(-1)\mathcal{T}$. (Thus the $\nabla_A{}^{B'}\chi$ is removed in the passage to $\hat{\mathcal{A}}$, and $\hat{\mathcal{A}}$ is the cotangent bundle.) As with $\mathbb{B}$, we have a bundle map

$$X : \mathcal{A} \longrightarrow \mathcal{T} \times \mathbb{C}$$

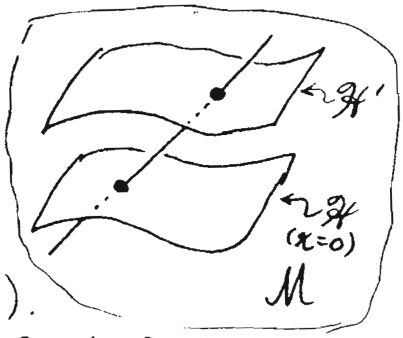
where

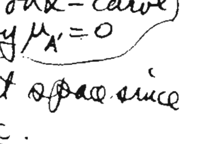
$$X(W_\alpha) = \chi + \mu^{A'}\pi_{A'} = \chi + W_\alpha Z^\alpha.$$

Z^α rep. by $(0, \pi_{A'})$ on α-plane

(On $\hat{\mathcal{A}}$ this gives $X = W_\alpha Z^\alpha$.) As with $\mathbb{B}$, $\mathcal{A}$ serves to define $\mathcal{H}$ within $\mathcal{M}(\mathcal{H})$, and we believe that $\mathcal{A}$ fixes Ω also (not yet checked!)

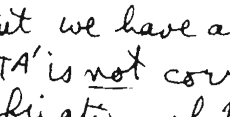
What we should like to do would be to evolve the Einstein equations and obtain $\mathcal{M}$. This would require something like using null geodesics defined by $W_\alpha Z^\alpha = 0$ (ambitwistors) to map from one $\mathcal{H}$ to another $\mathcal{H}'$ and to extend away from $W_\alpha Z^\alpha = 0$ (preferably to all orders, in some way). This would require our being able, given (W_α, Z^α) on $\mathcal{H}$, to find (W_α, Z^α) on $\mathcal{H}'$. Take $\mathcal{H}'$ to be infinitesimally above $\mathcal{H}$. What we need is a way of moving an α-curve from $\mathcal{H}$ to $\mathcal{H}'$, given a W_α on it. In the flat case, this movement is just that given by the α-plane in $\mathcal{M}$ (which now exists). The displacement is given by





connecting vectors (null) $\lambda_*^A \pi_*^{A'}$

$t_{AA'}\lambda_*^A\pi_*^{A'} = t_{AA'}t^{AA'}$



original pt. on α-curve on $\mathcal{H}$ given by $\mu_{A'} = 0$

If $W_\alpha Z^\alpha = 0$, we could start off the null geodesic defined by (W_α, Z^α) at the point on the α-curve on $\mathcal{H}$ where $\mu^{A'} = 0$, and then displace away along $\lambda^A\pi^{A'}$. We want this displacement to work also for the other points on the α-curve. This is O.K. in flat space since we can take the direction of $\lambda^A\pi^{A'}$ elsewhere on the α-curve to be constant.

In curved space it seems possible that the λ^A that we have already been using might still work (i.e. $\lambda_*^A = \lambda^A$), but the $\pi^{A'}$ is not correct (i.e. $\pi_*^{A'} \neq \pi^{A'}$, in general). We seem to need a modification of $\pi^{A'}$ in order for this to work elsewhere along the curve. So far we do not have a suitable primed spinor that varies along the α-curve in a way that is not proportional to $\pi^{A'}$. We need all this so that we can propagate (W_α, Z^α) when $W_\alpha Z^\alpha \neq 0$, and there is no null geodesic defined by (W_α, Z^α).

More work in progress.

L J Mason & Roger Penrose

An alternative form of the Ernst potential

James Fletcher

One of the strengths of the twistor description of the stationary axisymmetric solutions of Einstein's equations is the light it sheds on the relation between the metric and its Ernst potential. Recall that to represent a solution we have a bundle E over a reduced twistor space R_U which consists of two Riemann spheres S_0 and S_1 (with coordinate w) identified over the set U. The usual Ward construction gives us a solution of Yang's equation, usually denoted by J, which is the metric on the space of Killing vectors. If we impose the conditions that J be regular on the symmetry axis $r = 0$ and satisfy $\det J = -r^2$ then $E|_{S_0} = L_1 \oplus L_0$ and $E|_{S_1} = L_{-1} \oplus L_0$ where L_1 is the tautological bundle, L_{-1} is the hyperplane section bundle and L_0 is the trivial bundle. We can describe E by means of patching matrices $P_{\alpha\beta}$ defined on the overlaps of a collection of open sets U_α which cover R_U. If we take $U_0 \subset S_0$ and $U_1 \subset S_1$ to be neighbourhoods of $w = \infty$ not containing $w = 0$, and $U_2 \subset S_0$ and $U_3 \subset S_1$ to be neighbourhoods of $w = 0$ not containing $w = \infty$, then

$$P_{02} = \begin{pmatrix} 2w & 0 \\ 0 & 1 \end{pmatrix}, \quad P_{13} = \begin{pmatrix} (2w)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and E is completely specified by one of the patching matrices between the spheres, for example by P_{23} which I shall denote by P.

We can construct another solution $\iota(J)$ from the related bundle $\iota(E)$. To obtain $\iota(E)$, we take the same cover $\{U_\alpha\}$ and the same patching matrix P between the two spheres, but replace both P_{02} and P_{13} by the identity. Thus the restrictions $\iota(E)|_{S_0}$ and $\iota(E)|_{S_1}$ are both trivial; and we can think of the operations ι and ι^{-1} as untwisting and twisting the bundle round the points at infinity. We can write the corresponding matrix in the form

$$\iota(J) = \begin{pmatrix} f - \frac{\psi^2}{f} & \frac{\psi}{f} \\ \frac{\psi}{f} & \frac{1}{f} \end{pmatrix}$$

then the usual Ernst potential $\mathcal{E}$ is given by $\mathcal{E} = f + i\psi$. NMJW and LJM (1988) used this as the starting point for encoding the Geroch group in the twistor picture.

Instead of twisting and untwisting about $w = \infty$, however, we can do the same about $w = 0$. In other words, given our original bundle E, we can define a new one $\hat{E}$ which is also trivial over each of the spheres S_0 and S_1, but for which the patching matrix between them is P_{01} instead of P_{23}. Since P_{01} is given by

$$P_{01} = \begin{pmatrix} 2w & 0 \\ 0 & 1 \end{pmatrix} P \begin{pmatrix} 2w & 0 \\ 0 & 1 \end{pmatrix},$$

it has determinant equal to $(2w)^2$ and it is actually more convenient to use $P' = (1/2w)P_{01}$ to define the new bundle E'. I explained in my article in TN27 what it means for a patching matrix to be *adapted* to a certain part of the axis $r = 0$; if we suppose that P is adapted to an interval of the form $(0, a)$ then replacing P_{01} with P' corresponds to dividing the corresponding matrix J by u^2 to obtain J', where (in terms of the usual Weyl coordinates (z, r)), u and v are given by $r = uv$ and $z = \frac{1}{2}(u^2 - v^2)$.

Note that there is a certain amount of freedom in the construction of J'. With a particular choice of the two Killing vectors (X_1, X_2) in the original space-time (which must be arranged such that X_1 vanishes or is null on $r = 0$ in the interval $(0, a)$), we can still transform P by $P \mapsto BPC$, where

$$B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

for constants b and c. This will alter J' while leaving J the same. Going in the other direction, however, there is a unique P' for each J' and the only choice that can arise occurs if the twistor space $R_{U'}$ is glued down at $w = 0$; we then have to decide to which of the spheres S_0 and S_1 to assign each of the points at $w = 0$ in R_U.

There is a direct method for passing between J' and J which is analogous to changing from Ernst potential to metric. If we choose a Ward splitting $\{K'_\alpha\}$ for the patching matrices $P'_{\alpha\beta}$ describing E' such that

$$K'_2 = \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix} \text{ at } \lambda = -(v/u) \text{ and } K'_3 = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \text{ at } \lambda = u/v$$

(where λ is the coordinate on the CP^1 in PT corresponding to the orbit (u, v)) then, provided we have chosen S_0 and S_1 such that $w(u/v) \in S_0$ and $w(-v/u) \in S_1$, it follows that

$$J = H' \begin{pmatrix} -v^2 & 0 \\ 0 & u^2 \end{pmatrix} (\hat{H}')^{-1}$$

where $H' = K'_0(0)$ and $\hat{H}' = K'_1(\infty)$ (and so $J' = H'(\hat{H}')^{-1}$).

This choice of K'_2 and K'_3 corresponds to a choice of complex structure Ψ (in the Dolbeault version of the Ward construction: see NMJW & LJM (1986) such that

$$\Psi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \text{ at } \lambda = u/v \text{ and } \Psi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \text{ at } \lambda = -v/u. \qquad (1)$$

This is because, on the pull-back of the set U_α to PT, $\Psi = K_\alpha^{-1}\partial K_\alpha$.

It is straightforward to show (from the definition of Ψ) that (1) implies we have chosen $\hat{H}' = (\mathrm{s}_1\ \mathrm{s}_2)$ and $H' = J'\hat{H}'$,where s_1 and s_2 are solutions of the equations

$$\partial_r \mathrm{s} + \frac{1}{1+\lambda^2}((J')^{-1}\partial_r J' - \lambda(J')^{-1}\partial_z J')\mathrm{s} \;=\; 0 \qquad (2)$$

$$\partial_z \mathrm{s} + \frac{1}{1+\lambda^2}(\lambda(J')^{-1}\partial_r J' + (J')^{-1}\partial_z J')\mathrm{s} \;=\; 0 \qquad (3)$$

with $\lambda = u/v$ and $\lambda = -v/u$ respectively. To go in the other direction, we replace J' with $u^{-2}J$ and choose $\hat{H} = (\mathrm{s}_2\ \mathrm{s}_1)$.

There are two dimensions of freedom in the choice of s_1 and s_2 in each case; and in fact defining $\hat{H}'$ in this way only implies that K'_2 and K'_3 are of the form

$$K'_2 = \begin{pmatrix} * & \alpha \\ * & \beta \end{pmatrix} \quad \text{and} \quad K'_3 = \begin{pmatrix} \gamma & * \\ \delta & * \end{pmatrix}$$

where α and β are constant on $\lambda = u/v$, and γ and δ are constant on $\lambda = -v/u$. Defining $\hat{H}$ in a similar way gives a similar form for K_2 and K_3. In both cases, the behaviour of the splitting matrices on the two surfaces in PT is due to the fact that the only holomorphic functions on PT which are invariant under the lifts of the two Killing vectors are those which are functions of w alone, where w is related to λ by the equation

$$w = \frac{r}{2}(\lambda^{-1} - \lambda) + z.2) \qquad (4)$$

When we go in the 'twisting' direction (that is to say, from J' to J) we can actually fix s_1 and s_2, and thus J, completely by considering the behaviour of K'_2 and K'_3 as $v \to 0$. This corresponds to the unique choice of P' in this case. On the other hand, the freedom that we have in the other direction also corresponds precisely to the freedom in the choice of patching matrix P.

Finally, a brief remark on the point of all this. It turns out that if a space-time has both a symmetry axis and a Killing horizon and is regular at the point where they intersect, then the patching matrix P has a simple pole at the point in the reduced twistor space which corresponds to the intersection and which we can assume to be at $w = 0$ (see JF & NMJW 1990). It is straightforward to show, however, that the 'untwisted' patching matrix, P', is well-behaved on the real axis near $w = 0$, and slightly less straightforward to show that its entries are actually holomorphic in a neighbourhood of this point.

References

NMJW & LJM Nonlinearity **1** 73 - 114 (1988) JF in TN **27** 14 - 16 JF & NMJW in 'Reviews in Twistor Theory' ed RJB & TNB (1990)

Thanks to NMJW and to Fletcher and Partners

James Fletcher

Structure of the jet bundle for manifolds with conformal or projective structure

Let G be a complex semi-simple Lie group and P<G a Lie subgroup. To each P-module E there corresponds a homogeneous vector bundle $\boldsymbol{E}$, over G/P, which is just G$\times E$ modulo the equivalence relation $(\mathrm{g},e) \sim (\mathrm{gp}, \mathrm{p}^{-1}e)$ for g$\in$G,p$\in$P and $e \in E$. We write

$$\boldsymbol{E} = \mathrm{G}\times_{\mathrm{P}} E.$$

It is an interesting exercise (see [2]) to check that the the bundle of infinite jets, $\boldsymbol{JE}$, associated with this homogeneous bundle is itself homogeneous:

$$\boldsymbol{JE} = \mathrm{G}\times_{\mathrm{P}} JE$$

where JE may be obtained as the fibre of $\boldsymbol{JE}$ over the identity coset P$\in$G/P.

It is interesting and useful to see the extent to which this goes through in case G/P is replaced by a more general structure. So let $\mathcal{M}$ be an arbitrary complex holomorphic manifold equipped with a P-principal bundle $\mathcal{G}$. Then $\mathcal{M}$ is the fibrewise quotient $\mathcal{G}/\mathrm{P}$ and we have an appropriate generalisation of the above situation. Corresponding to the homogeneous bundles above are *semi-homogeneous* bundles which are constructed in exactly the same way: If E a P-module then we have $\boldsymbol{E} := \mathcal{G} \times_{\mathrm{P}} E$.

A section f of $\boldsymbol{E}$ corresponds to a function F,

$$F : \mathcal{G} \to E,$$

such that

$$\mathrm{p}F(g\mathrm{p}) = F(g) \quad \text{for} \quad g \in \mathcal{G} \text{ and } \mathrm{p} \in \mathrm{P}.$$

Such functions F will be said to be *semi-homogeneous.* The space of such E-valued semi-homogeneous functions is itself a P-module in the obvious way. This has a P-submodule of semi-homogeneous functions which vanish to order $k+1$ on $g\mathrm{P}$ for $x = g\mathrm{P}$ an arbitrary point of $\mathcal{M}$. The quotient P-module shall be denoted $J^k_x E$. Points of the bundle of k-jets associated to $\boldsymbol{E}$, over x, correspond to points in $g\mathrm{P}\times J^k_x E$ modulo the equivalence relation $(g, F) \sim (g\mathrm{p}, \mathrm{p}^{-1}F)$ for $\mathrm{p} \in \mathrm{P}$ and $F \in J^k_x E$. Thus we see that the bundle of k-jets, $\boldsymbol{J^k E}$, has an underlying P-structure as in the homogeneous case above. However in this more general setting the inducing P-module may vary from point to point of $\mathcal{M}$. We could write

$$\boldsymbol{J^k E} = \mathcal{G} \times_{\mathrm{P}} J^k_{g\mathrm{P}} E$$

to describe this. With the same notation the bundle of infinite jets is given by

$$\boldsymbol{JE} = \mathcal{G} \times_{\mathrm{P}} J_{g\mathrm{P}} E,$$

where, for each $g\mathrm{P}\in \mathcal{G}/\mathrm{P}$, $J_{g\mathrm{P}}E$ is the projective limit over k of the $J^k_{g\mathrm{P}}E$.

The dual version of this proceeds as follows. Let $\mathcal{D}$ denote the space of differential operators from E-valued functions on $\mathcal{G}$ to $\mathbf{C}$-valued functions on $\mathcal{G}$. When restricted to act on semi-homogeneous functions, F, $\mathcal{D}$ gains a P-module structure: For $D \in \mathcal{D}$, pD is defined by

$$[\mathrm{p}D]F(g) := DF(\tilde{g})|_{\tilde{g}=g\mathrm{p}} = D\mathrm{p}^{-1}F(\tilde{g}\mathrm{p}^{-1})$$

where $g \in \mathcal{G}$, p$\in$P and on the extreme right hand side $F(\tilde{g}p^{-1}) = F(g)$ is to be regarded as a function of $\tilde{g}$. $\mathcal{D}$ has a P-submodule of operators which act as zero on semi-homogeneous functions. Denote by $\tilde{X}(E)$ the quotient P-module. Let $\mathcal{O}_\mathcal{M}$ denote the C-valued semi-homogeneous functions on $\mathcal{G}$, i.e., the functions constant on each fibre gP, $g \in \mathcal{G}$. For some fixed $x \in \mathcal{M}$, let $\mathcal{I}_x < \mathcal{O}_\mathcal{M}$ be the subspace of functions which vanish over x. Denote by $\mathcal{I}_x.\tilde{X}(E)$ the P-submodule of $\tilde{X}(E)$ which consists of elements of $\tilde{X}(E)$ left multiplied by functions in $\mathcal{I}_x$. Since $\tilde{X}(E)$ is naturally a $\mathcal{O}_\mathcal{M}$-module it is clear that $\mathcal{I}_x.\tilde{X}(E)$ is a P-submodule of $\tilde{X}(E)$. Once again we can form the quotient module which we shall denote $\tilde{V}_x(E)$. This P-module is filtered naturally by order of the operators involved; write $\tilde{V}_{k,x}(E)$ to denote the submodule consisting of operators of order no greater than k. Since each element of $\tilde{V}_{k,x}(E)$ determines a map

$$J^k E \to \mathbf{C},$$

and $\tilde{X}(E)$ consists of all non-trivial operators on E-valued semi-homogeneous functions it is at once clear that $\tilde{V}_{k,x}(E)$ is precisely the vector dual of $J^k_x E$. It is easily checked that it is also dual as a P-module. Evidently then,

$$(\boldsymbol{J}^k \boldsymbol{E})^* = \mathcal{G} \times_\mathrm{P} \tilde{V}_{k,g\mathrm{P}}(E)$$

with notation understood to be as above.

While this and the dual version first mentioned provide a description of the jet bundle the situation is less than ideal. Even at the level of k-jets, since the inducing module is point dependent there is little scope for reducing the problem of finding differential operators of order $\leq k$ to a finite dimensional one. Nevertheless without more structure this is probably as far as one can go. However in many instances such structure is readily available ....

For example if $\mathcal{M}^n$ has a conformal (or projective) structure then one obtains a principal P-bundle, $\mathcal{G}$, where P is a particular parabolic subgroup of Spin$(n+2)$ (SL$(n+1)$ respectively). Moreover the bundle $\mathcal{G}$ comes equipped with a canonical notion of horizontality called the normal conformal (resp. projective) Cartan connection. We shall see that in either of these cases the jet bundle is almost as simple to describe as in the homogeneous case. The Cartan connection (which will always refer to the *normal* version) is usually described by a 1-form ϑ which satisfies (where $\mathbf{p}$ and $\mathbf{g}$ are the Lie algebras of P and G respectively):

(i) $\vartheta_q : T_q\mathcal{G} \to \mathbf{g}$ is an vector space isomorphism $\forall q \in \mathcal{G}$.

(ii) $\vartheta(\mathrm{X}^*_q) = \mathrm{X}$ if X* is the Killing field corresponding to X$\in \mathbf{p}$.

(iii) $R^*_\mathrm{p}\vartheta = \mathrm{Ad}(\mathrm{p}^{-1})\vartheta$, where R_p describes the right action of p$\in$P on $\mathcal{G}$.

as well as some curvature conditions. Note that if we write $\mathbf{g}^* := \vartheta^{-1}(\mathbf{g})$ then, regarding the vector fields $\mathbf{g}^*$ as differential operators, (iii) is equivalent to

$$[\mathrm{XY}]^* = [\mathrm{X}^*, \mathrm{Y}^*]$$

for arbitrary X$\in$ $\mathbf{p}$ and Y$\in$ $\mathbf{g}$. We can, in the obvious way, extend ϑ^{-1} to act on the tensor algebra, $\bigotimes \mathbf{g} := \bigoplus_{k=0}^\infty \otimes^k \mathbf{g}$. The result of this is a space of special differential operators on $\mathcal{G}$ which will be denoted by $\mathcal{U}(\mathbf{g}^*)$. There is a natural filtration of $\mathcal{U}(\mathbf{g}^*)$ induced from the grading of the tensor algebra $\bigotimes \mathbf{g}$; i.e., $\mathcal{U}_k(\mathbf{g}^*)$ is the image of $\bigoplus_{i=0}^{k} \otimes^i \mathbf{g}$. We note that $\mathcal{U}(\mathbf{g}^*)$ is strictly contained in $\mathcal{D}$, in fact $\mathcal{U}_k(\mathbf{g}^*)$ is finite dimensional.

The left $\mathcal{U}(\mathbf{g}^*)$-module

$$\mathcal{U}(\mathbf{g}^*) \otimes E^*$$

may be thought of as a special class of differential opertors from E-valued functions on $\mathcal{G}$ to $\mathbf{C}$-valued functions on $\mathcal{G}$. As operators restricted to semi-homogeneous functions we may consider the action of P (as described above for all of $\mathcal{D}$) on this space. Now we may regard this $\mathcal{U}(\mathbf{g}^*)$-module as a $\mathbf{p}^*$-module (or equivalently a $\mathbf{p}$-module) by restriction and it is readily verified that this agrees precisely with the P-action (at least treating the elements of $\mathcal{U}(\mathbf{g}^*) \otimes E^*$ as differential operators on semi-homogeneous functions). Thus $\mathcal{U}(\mathbf{g}^*) \otimes E^*$ is closed under this P-action and so, given property (i) of ϑ, is an ideal candidate to replace $\mathcal{D}$.

$\mathcal{U}(\mathbf{g}^*) \otimes E^*$ has a P-submodule of operators which annihilate all semi-homogeneous functions. Let $X(E)$ be the quotient and $\mathcal{O}_{\mathcal{M}}.X(E)$ consist of elements of $X(E)$ left multiplied by functions from $\mathcal{O}_{\mathcal{M}}$. Then $\mathcal{O}_{\mathcal{M}}.X(E)$ is also a P-module and, for any fixed $x \in \mathcal{M}$, has $\mathcal{I}_x.X(E)$ as a P-submodule. Again we form the quotient and denote the resulting P-module $V_x(E)$. With similar reasoning to that in used in the $\tilde{V}_x(E)$ case it is not difficult to see that $V_x(E) = (J_xE)^*$ and that

$$\mathcal{G} \times_{\mathrm{P}} V_{gP}E \equiv (\boldsymbol{J}\boldsymbol{E})^*.$$

In this construction also, the inducing P-module varies on $\mathcal{M}$. Thus at first glance it would seem that we are no better off than with the construction that began with $\mathcal{D}$. In fact, however, we now have a considerably more rigid structure as consideration at the level of k-jets reveals.

Write $\mathcal{D}_k < \mathcal{D}$ to mean the subspace of differential operators of order $\leq k$. Corresponding to this $\tilde{X}(E)$ will inherit a filtration, by $\tilde{X}_k(E)$ say. In the approach that begins with all differential operators, this is the key P-module leading to the construction of the dual k-jet bundle. The problem is that $\tilde{X}_k(E)$ is infinite dimensional and we know nothing about its structure. If a conformal or projective structure is present then corresponding to this one has $X_k(E)$, where the filtration of $X(E)$ by the $X_k(E)$ arises from the filtration of $\mathcal{U}(\mathbf{g}^*)$ by $\mathcal{U}_k(\mathbf{g}^*)$. Now, in contrast to $\tilde{X}_k(E)$, $X_k(E)$ is finite dimensional. In fact $X_k(E)$ looks just like certain Verma modules which arise in the homogeneous case with some modification due to the curvature of the Cartan connection. Thus, although the structure of $V_{k,x}(E)$ varies over $\mathcal{M}$, the variation involved is a relatively minor detail involving the actual value of the curvature at each point. The important point, however, is that we have a natural bundle epimorphism from a finite dimensional semi-homogeneous bundle onto $(\boldsymbol{J}^k\boldsymbol{E})^*$:

$$\mathcal{G} \times_{\mathrm{P}} X_k(E) \to (\boldsymbol{J}^k\boldsymbol{E})^*.$$

There is an immediate application of this result. Suppose there is a P-module monomorphism

$$i : H^* \to X_k(E).$$

Then this induces an invariant homomorphism of the corresponding semi-homogeneous bundles:

$$\mathcal{G} \times_{\mathrm{P}} H^* \to \mathcal{G} \times_{\mathrm{P}} X_k(E).$$

and thus a vector bundle homorphism,

$$\mathcal{G} \times_{\mathrm{P}} H^* \to (J^k \boldsymbol{E})^*.$$

Dually then, we have a bundle homomorphism

$$\boldsymbol{J^k E} \longrightarrow \boldsymbol{H},$$

that is, a differential operator; here $\boldsymbol{H}$ is of course $\mathcal{G} \times_{\mathrm{P}} H$. Beginning with irreducible modules H, finding injections such as i above is straightforward (in principle at least) and just involves finding certain vectors in $X(E)$ which are annihilated by a special subalgebra of $\mathbf{p}$. (These are called *maximal* vectors.) It is thus easy to see that for irreducible H and any k there are a finite number of differential operators which arise in this fashion. Indeed beginning also with E irreducible, the resulting operators are, in a real sense, analogues of the *invariant* operators in the homogeneous case (as in [2,1]) or composites thereof.

Examples of applications of these ideas can be found in [3].

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The geometry of the space of null geodesics

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Abstract

The topology and geometry of the space of null geodesics N of a space-time M are used to study the causal structure of the space-time itself. In particular, the question of whether the topology of N is Hausdorff or admits a compatible manifold structure carries information on the global structure of M, and the transversality properties of the intersections of skies of points tell whether the points are conjugate points on a null geodesic.

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A note on causal relations and twistor space

Let $\mathbf{M}$ represent Minkowski space, and $\mathbf{PN}^{\mathbf{I}}$ represent projective null twistor space without $\mathbf{I}$, the twistor line at infinity, regarded as a subset of $\mathbf{CP}^3$.

Lemma *If $x, y \in \mathbf{M}$, then $x \in J(y)$ if and only if there is a null curve in $\mathbf{M}$ joining x to y. In addition, $x \in I(y)$ if and only if this curve is not a geodesic.*
Proof Consider any null curve starting at x. This stays in $J(x)$ and enters $I(x)$ iff it is not a geodesic. Any point in $I(x)$ may be carried to any other by a Poincaré transformation fixing x. Such transformations do not alter the null or geodesic properties of a curve. □

If $\gamma : [0,1] \to \mathbf{M}$ is a smooth curve, denote by $\Gamma(t)$ the sky in $\mathbf{PN}^{\mathbf{I}}$ of $\gamma(t)$. The curve then gives a ruled surface in $\mathbf{PN}^{\mathbf{I}}$ denoted by $\tilde{\Gamma}$ and defined by $\{\Gamma(t) : t \in [0,1]\}$

Lemma *The surface $\tilde{\Gamma}$ is a developable if and only if γ is null.*
Proof A developable may be characterised as a ruled surface whose infinitesimally separated generators intersect (see, for example, Semple and Roth, *Introduction to algebraic geometry*, pp 255 ff). Now, neighbouring generators of $\tilde{\Gamma}$ are the skies of neighbouring points on γ and hence correspond to null separated points; they therefore intersect. □

Using these two, and observing that $\tilde{\Gamma}$ has self intersections only when two points of γ are null separated, one can show that the following two results hold.

Proposition *$x \in I(y)$ if and only if $X \cup Y$ is the boundary of a developable with no self intersections.* □

Corollary *Let Σ be a ruled surface in $\mathbf{PN}^{\mathbf{I}}$ with boundary $X \cup Y$. Then $x \in I(y)$ if and only if Σ can be deformed into a developable.* □

From these results we see that the causal nature of the interval between two points in $\mathbf{M}$ admits of a fairly direct interpretation in $\mathbf{PN}^{\mathbf{I}}$ in terms of its projective geometry. Note that it is important that we do not use $\mathbf{PN}$, as this would allow us to find a (non self-intersecting) developable with boundary $X \cup Y$ for any pair of points not lying on a single null geodesic.

Robert Low

Harmonic morphisms and mini-twistor space.

A harmonic morphism is a map $\varphi: M \to N$ of Riemannian manifolds M, N with the following property: *$f: N \to R$ is harmonic iff $\varphi \circ f: M \to R$ is.* As a concrete example, take M to be $\mathbf{R}^3$ with coordinates x, y, z and N to be $\mathbf{R}^2$ with coordinates u, v. The map φ is defined by giving $u(x,y,z)$, $v(x,y,z)$ satisfying

$$\nabla^2 u = \nabla^2 v = 0 = \nabla u \cdot \nabla v \ ; \ |\nabla u|^2 = |\nabla v|^2 \qquad (1).$$

In this case Baird and Wood [**BW**] find that φ is locally defined by a holomorphic curve in TP1, the tangent bundle of the complex projective line. They go on to use this fact to classify globally defined harmonic morphisms in this case, and also in the cases $\mathbf{S}^3 \to$ surface and $\mathbf{H}^3 \to$ surface.

Since TP1 is the mini-twistor space of $\mathbf{R}^3$ it is natural to wonder what, if anything, is the relation to twistor theory of this property of φ. In the case when dim M = 3, dim N = 2, the inverse images of points of N give curves in M. One purpose of this note is to observe that *the defining property of harmonic morphisms is equivalent to the condition that this congruence of curves be a geodesic and shear-free congruence.*

Now TP1 is the space of geodesics of the flat metric on $\mathbf{R}^3$ and so a congruence of geodesics corresponds to a 2-real parameter surface in TP1. As one might anticipate from the Kerr theorem, there is a mini-Kerr theorem that

this surface is a holomorphic curve iff the congruence is shear-free.

In particular, this leads to an explicit formula for such congruences: if the generator is

$$L = \frac{1-\alpha\alpha}{1+\alpha\alpha}\frac{\partial}{\partial z} + \frac{\alpha+\alpha}{1+\alpha\alpha}\frac{\partial}{\partial x} - \frac{i(\alpha-\alpha)}{1+\alpha\alpha}\frac{\partial}{\partial y}$$

then $\alpha(x,y,z)$ is given implicitly by

$$f(x(1-\alpha^2) + iy(1+\alpha^2) + 2\alpha z, \alpha) = 0 \quad \text{or in spinors} \quad F(x^{AB}\alpha_A\alpha_B, \alpha_C) = 0$$

for arbitrary holomorphic f or holomorphic and homogeneous F (a formula similar to this is in [**BW**].

As Baird and Wood remark, to find solutions of (1) was set as a problem by Jacobi. This now falls into the class of non-linear differential-geometric problems solvable by twistor theory.

I am grateful to John Wood and Paul Baird for telling me about harmonic morphisms.

BW Baird and Wood 1988 Math. Ann. **280** 579-603
see also Baird 1987 Ann. Inst. Fourier, Grenoble 37 135-173
Baird and Wood *Harmonic morphisms and conformal foliations by geodesics of three-dimensional space-forms* University of Melbourne Department of Mathematics Research Report no.2-1989

KPT

Twistors and SU(3) monopoles

Hitchin [1] has shown that SU(2)-monopoles of charge k on $\mathbb{R}^3$ are equivalent to algebraic curves (spectral curves) of genus $(k-1)^2$, satisfying certain constraints, lying in the minitwistor space $T\mathbb{P}^1$. Now $T\mathbb{P}^1 = \{(\underline{u}, \underline{v}) : \underline{u}, \underline{v} \in \mathbb{R}^3; \|\underline{u}\| = 1, \underline{u}.\underline{v} = 0\}$ so it may be identified with the space of oriented lines in $\mathbb{R}^3$. Also $T\mathbb{P}^1$ fibres over $\mathbb{P}^1$ and we may take coordinates (η,ζ) on $T\mathbb{P}^1$ where ζ is a coordinate on $\mathbb{P}^1$ and η is a fibre coordinate. There is a real structure on $T\mathbb{P}^1$; in terms of the above coordinates it is $T:(\eta,\zeta) \mapsto (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$, but it is easier to think of it as just reversing the orientation of oriented lines in $\mathbb{R}^3$. We define line bundles L^t of degree 0 over $T\mathbb{P}^1$ by letting L^t be the bundle with transition function $\exp(t\eta/\zeta)$.

For each SU(2)-monopole there is just one associated spectral curve S in $T\mathbb{P}^1$. It satisfies:-

(i) S is compact and has equation $\eta^k + a_1(\zeta)\eta^{k-1} + \ldots + a_k(\zeta) = 0$ where each a_i is a polynomial of degree $2i$

(ii) L^2 is trivial over S; or equivalently (since $\deg L^2 = 0$), $H^0(S,L^2) \neq 0$

(iii) S is preserved by the real structure τ

(iv) S has no multiple components

(v) (nondegeneracy condition) $H^0(S,L^t(k-2)) = 0$ for $0 < t < 2$

A parameter count gives the dimension of the moduli space of charge k SU(2)-monopoles as 4k-1.

These results have been extended to the case of SU(n)-monopoles with symmetry broken to U(1)x...x U(1) by Michael Murray [3] who showed that such monopoles were generically determined by n-1 spectral curves (satisfying certain constraints) in minitwistor space.

We may also consider monopoles with nonmaximal symmetry breaking i.e. symmetry broken to a nonabelian subgroup. In particular consider SU(3) monopoles with symmetry to U(2). As SU(3) is the QCD gauge group such monopoles may be of particular physical interest.

We now have only one spectral curve (as opposed to two curves for U(1) x U(1) symmetry breaking). This curve satisfies conditions (i) and (iii) above; however the condition that there is a nontrivial element of $H^0(S,L^2)$ is replaced by the requirement that $H^0(S,L^3(l,l)) \neq 0$ (for the charge $2l$ monopole). Parameter counting, using results from algebraic geometry about the dimension of linear systems on algebraic curves, suggests that the charge $2l$ moduli space should have dimension $\leq 12l-1$; in fact the charge 2 moduli space should have dimension precisely 11 (or 8 once we fix the centre of the monopole in $\mathbb{R}^3$). This agrees with a result of E. Weinberg [5] (Weinberg includes an S^1 phase to get 12 parameters).

Further investigations concerning nondegeneracy conditions suggest that the 7-dimensional space of SU(2) charge 2 monopoles should arise as a boundary of the 8-dimensional space of SU(3) minimal symmetry breaking charge 2 monopoles. Now it is known [2] that SU(2) charge k monopoles are equivalent to triples.

(T_1, T_2, T_3) of k x k-matrix valued functions on [0,2] satisfying:-

(1) $T_i^*(t) = -T_i(t)$

(2) $T_i(2-t) = -T_i(t)$

(3) T_i is analytic on (0,2) with simple poles at $t = 0,2$

(4) $\frac{dT_1}{dt} = [T_2, T_3]$ and cyclically (Nahm's Equations).

(5) The residues of the T_i at $t = 0,2$ give an irreducible representation of SU(2).

The pole at $t=2$ corresponds to the bundle L^2 being trivial over the spectral curve S. Condition (2) reflects the quaternionic nature of SU(2) ($\cong$ Sp(1)). In the SU(3) case, therefore, we should drop these conditions. The resulting modified system of Nahm's equations may be solved (in the charge 2 case) explicitly using SO(3) and SU(2) symmetries and Jacobi elliptic functions. The moduli space of centred SU(3) charge 2 monopoles is then an 8-dimensional space with the SU(2) moduli space as a boundary. The moduli space includes a spherically symmetric monopole and a 3-parameter family of axisymmetric monopoles; this agrees with results of Ward arrived at via twistor theory [4]; (Ward considers uncentred monopoles and so gets a 6-parameter family of axisymmetric solutions).

A.S. Dancer

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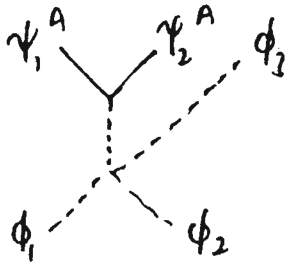
Twistor translation of Feynman vertices

Progress has been made in the programme of translating general Feynman diagrams into twistor diagrams. The advance comes through observing the *breaking of conformal invariance* in elementary Feynman amplitude calculations (that is, conformal symmetry-breaking even before renormalisation considerations come in). An example shows this explicitly:

Consider the Feynman diagram



where dashed lines indicate scalar fields and solid lines spin-1/2 fields. Such a diagram arises in the standard model from the presence of terms in the interaction Lagrangian of form

$$\psi_A \bar{\psi}^A \phi, \qquad (\phi\bar{\phi})^2$$

The Feynman diagram as drawn indicates the integral

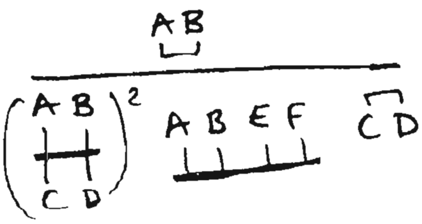
$$\int d^4x\, d^4y\; \psi_1^A(x)\,\psi_{2A}(x)\,\Delta_F(x-y)\,\phi_1(y)\,\phi_2(y)\,\phi_3(y)$$

considered as a functional of the external fields, where ϕ_1, ϕ_2 are of positive frequency and all the others of negative frequency.

We now choose particular fields, specified by corresponding twistor 1-functions:

$$\psi_1^A(x) \leftrightarrow \frac{1}{(W\cdot C)(W\cdot D)^2}, \quad \psi_2^A(x) \leftrightarrow \frac{1}{(W\cdot C)^2(W\cdot D)}, \quad \phi_1 = \phi_2 \leftrightarrow \frac{1}{(A\cdot Z)(B\cdot Z)}, \quad \phi_3 \leftrightarrow \frac{1}{(E\cdot Z)(F\cdot Z)}$$

Space-time calculation then yields the result of the Feynman integral, namely



This is $I^{\alpha\beta}$-dependent although scale-invariant (of homogeneity 0 in $I^{\alpha\beta}$.)

Hence we know that any proposed twistor diagram for this amplitude must involve $I^{\alpha\beta}$ in some way. Twistor diagrams therefore cannot hope to give a manifestly conformally invariant description of amplitudes in general: they can however make the dependence on $I^{\alpha\beta}$ explicit.

We can now arrive at the same conclusion by a different and more general argument. Note that the relation

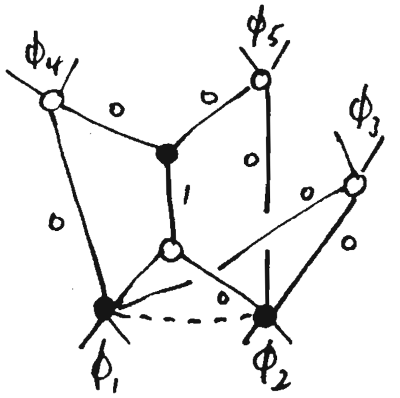
$$\iint d^4x\, d^4y \left(\Box_x \psi_1{}^A(x)\psi_{2A}(x)\right)\Delta_F(x-y)\,\phi_1(y)\,\phi_2(y)\,\phi_3(y)$$

$$= \int d^4x\, \psi_1{}^A(x)\psi_{2A}(x)\,\phi_1(x)\,\phi_2(x)\,\phi_3(x)$$

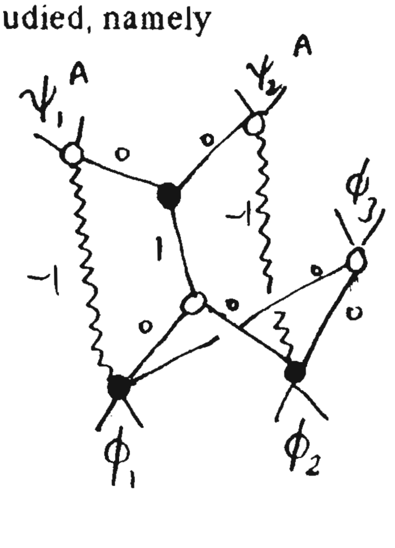
means that any proposed twistor representation of the Feynman integral above must be related via a differential operator with a representation of

the integral $\int d^4x\ \phi_4(x)\,\phi_5(x)\,\phi_1(x)\,\phi_2(x)\,\phi_3(x)$

Now there are many possible representations of this "ϕ^5" integral. One of them is



and this one allows the inverse of this differential operator to be applied, at least formally. This yields a candidate twistor diagram for the Feynman integral being studied, namely



But is there a contour for this diagram yielding the Feynman amplitude? NO. Proof by contradiction: suppose there were, then we could use it to effect an integration by parts of the "ϕ^5" diagram. But now take the limit as $\phi_5(x)$ moves towards the constant field, i.e the elementary state based at $I^{\alpha\beta}$. If the integration by parts were valid, this limit would be zero identically. But this limit must in fact be the integral

$$\int d^4x\, \phi_1(x)\, \phi_2(x)\, \phi_3(x)\, \phi_4(x)$$

which is non-zero in general.

Clearly this argument is not specific to this diagram and can be applied in general in situations where there are more than two in- or out-fields.

But suppose we allow ourselves to add further elements to the integral which involve $I^{\alpha\beta}$ explicitly in the singularity structure. Then the argument above can no longer be used to yield a contradiction. The limiting case of the constant field cannot taken since the contour may pinch in this limit.

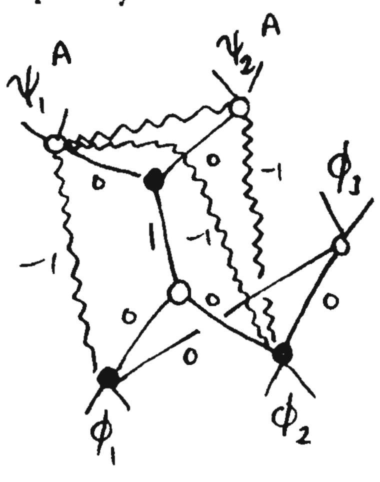
In fact there is a natural candidate for the form to be taken by these new elements, namely *boundaries at infinity*, i.e. boundaries on subspaces

$$\{ \overline{XZ} = 0 \} \quad \text{or} \quad \{ \underline{WY} = 0 \}$$

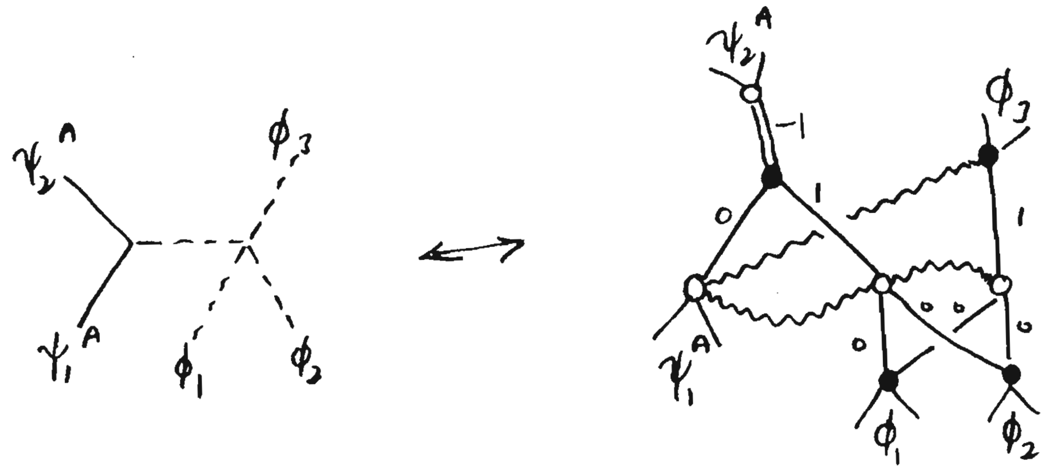
represented in diagrams by 

Note that such boundaries are "invisible to" the ∇, $\Box$ operators, which is why it is permissible to add them without upsetting the essential differential equation satisfied by (2).

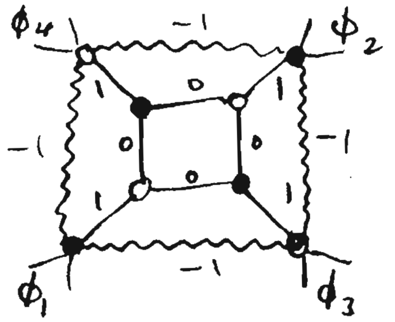
In fact one finds explicitly that



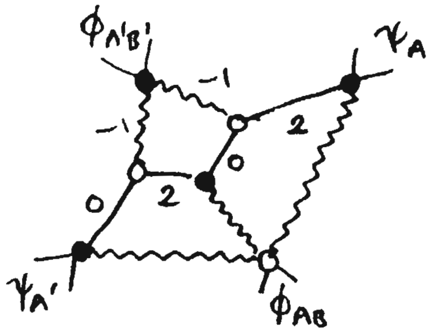
can be integrated to yield the correct amplitude. Similarly we may consider the channel in which the spin-1/2 fields have are of opposite frequency types and find a correspondence



We are now faced with the problem, familiar from earlier work, of finding a twistor interpretation of the crossing relations. In **TN 28**, analysis of the double-box led to the observation that the diagram



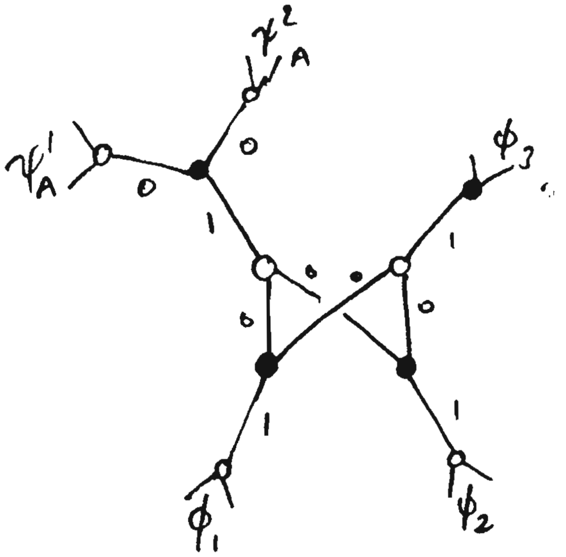
contained all channels for first-order ϕ^4 scattering within it; similarly for the diagram



with respect to Coulomb scattering.

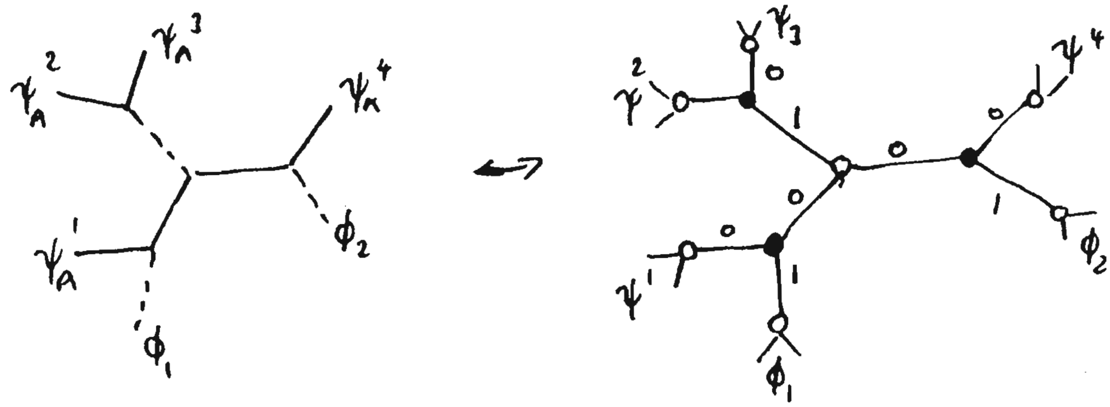
We might rephrase this observation in terms of asserting the existence of a *skeleton diagram* formed solely out of pole singularities, defining a twistor differential form which when combined with an appropriate collection of boundary prescriptions can yield the amplitude in any of the crossing-related channels.

Skeleton diagrams can be given for all the (2 $\to$ 2) processes hitherto studied. We now notice that the diagrams (3) and (4) above can be regarded in this sense as different realizations of the skeleton diagram



It has not yet been checked in detail that boundary-contours exist to yield *all* channels, but this seems very probable.

Attempted generalizations are now readily suggested. In particular, it appears that the process involving three "Yukawa interaction" vertices has a correspondence with a skeleton diagram given by



(This is certainly valid for *some* channels).
The significance of this process lies in regarding the external fields as test functions for the product of three Feynman propagators. Thus, looking at this diagram in all possible channels is equivalent to giving the full information of a general Feynman diagram vertex. Composition of these vertices into general Feynman diagrams should then be possible.

At present there is no indication of how the boundaries corresponding to each channel may be defined (they are certainly not uniquely defined.) Thus we are still faced with the problems of definition that have always arisen in diagram theory. However, this skeleton diagram concept at last suggests a framework which can include everything known so far and has the potential for systematic generalization.

Clearly we shall in general have to modify the definitions to introduce *inhomogeneous* poles and boundaries - possibly also logarithmic factors rather than boundaries - if we are to incorporate the inhomogeneity which eliminates infra-red divergences at first-order level. Note that to make the *boundaries at infinity* inhomogeneous would imply introducing a *mass* parameter.

There are then at least three directions which the existing results suggest for investigation:

(1) establishing the general vertex for Yukawa and ϕ^4 interaction, and then studying **Feynman loop diagrams** within these theories. Can the introduction of inhomogeneity eliminate ultra-violet divergences systematically?

(2) generalising these higher order calculations to **massless electroweak** theory. As may be seen from the Coulomb "skeleton", the feature that seems to be emerging is that gauge fields appear in the boundary specifications and not in the skeleton. This suggests the hopeful picture of a twistor calculus in which gauge fields are all absorbed into geometry in a manifestly gauge invariant way. In particular, for *pure* gauge field scattering the skeleton should virtually disappear, leaving only the specification of a bounded region of integration. This might give the closest link between diagram theory and the conformal field theory picture.

(3) The primary significance of the Yukawa interaction in the standard model is that it provides the mechanism for **massive fields** to arise. The essential idea is that the scalar field is, in the zeroth order, the constant field. The contours for the twistor diagrams we have written down will in general "pinch" as the external field tends towards the constant field (i.e. the elementary state based at $I^{\alpha\beta}$) but by studying this limit it may be possible to find a modification of twistor geometry at $I^{\alpha\beta}$ which allows the massive fields to emerge as a finite limit. The massive fields have already been described by twistor integrals involving inhomogeneous *poles* at infinity and it should be possible to relate these to the suggested formalism.

The combination of all three directions of generalization would amount to a general theory for translating Feynman diagrams into twistor diagrams.

A.P.H.

Spin Networks and the Jones Polynomial

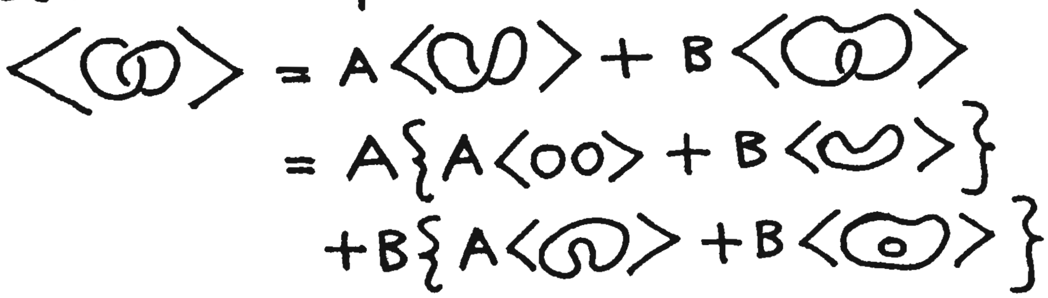
by Louis H. Kauffman

The purpose of this note is to point out that <u>the binor calculus of Roger Penrose is a special case of the Jones polynomial</u>, and that this generalization naturally involves the movement from SL(2) invariant tensor diagrams to SL(2)q invariant tensor diagrams, where SL(2)q denotes the quantum group!

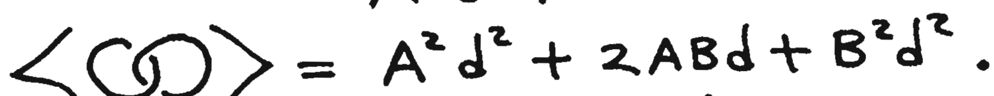
In order to see this, I first recall the bracket states model for the Jones polynomial [3], [4], [2]. We are given a function $\langle K\rangle$ defined on un-oriented link diagrams such that $\langle K\rangle$ is a polynomial in A, B and d. We assume that

and $\langle O \rangle = d$, $\langle O K\rangle = d\langle K\rangle$.

The small diagrams are parts of larger diagrams, identical except as indicated. For example,

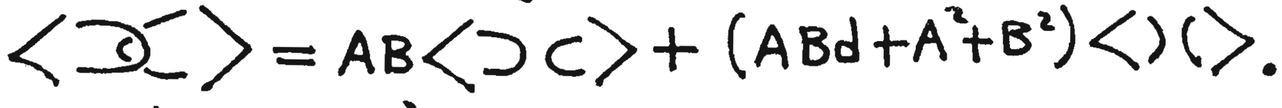


$$= A^2d^2 + ABd + BAd + B^2d^2$$



(A, B and d commute.)

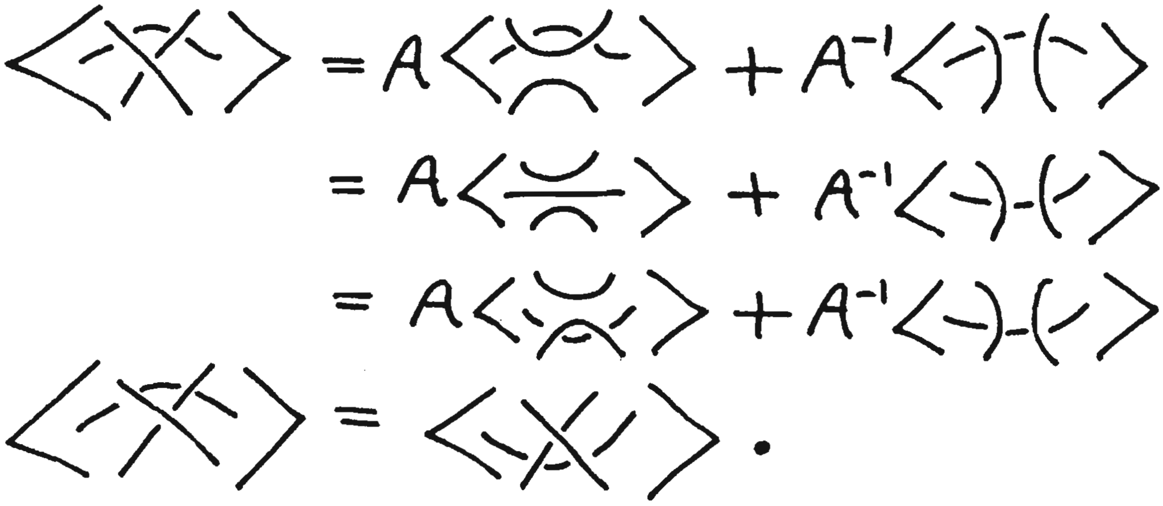
To create a topological invariant, we use the following formula:



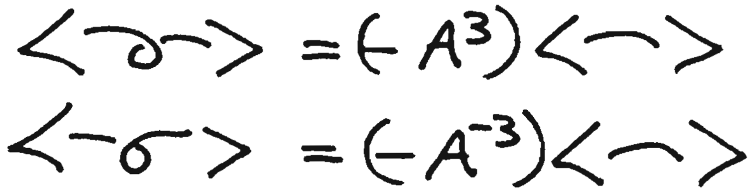
(easily proved)

Thus *if* $B = A^{-1}$ and $d = -A^2 - A^{-2}$

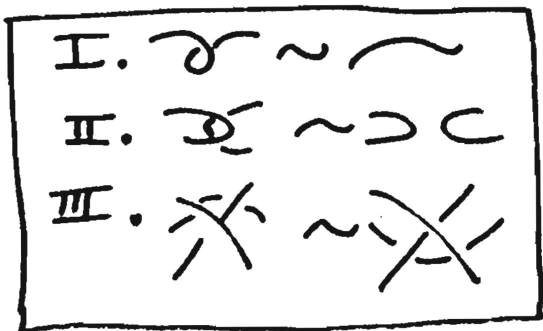
then  . It then follows directly that



Finally,



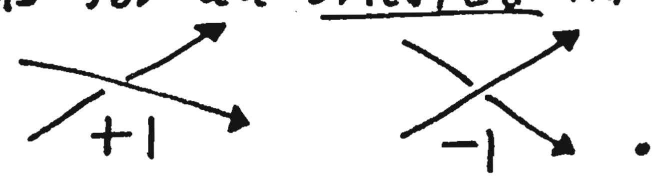
and so $\langle K \rangle$ is an invariant of Reidemeister moves II and III, and it is well-behaved on I.



These moves generate topological equivalence of knots and links.

REIDEMEISTER MOVES

By suitable normalization, one gets the [2] Jones polynomial: $V_K(t) = f_K(t^{-1/4})$ where $f_K(A) = (-A^3)^{-w(K)} \langle K \rangle / \langle O \rangle$ and $w(K)$ is the sum of the crossing signs for an *oriented* link K:



Now we shall stay with the bracket, and observe that if $A=-1$, then

1. $\langle \times \rangle = -\langle \asymp \rangle - \langle)(\rangle = \langle \times \rangle$
2. $\langle O \rangle = -2$.

Thus <u>bracket at $A=-1$ reproduces the binor formalism</u>: $\boxed{\begin{array}{c} \times + \asymp +)(= 0 \\ O = -2 \end{array}}$

Now the binors were created to be topologically well-defined $SL(2)$ invariant tensor diagrams. In particular, we can let $\underset{a\,b}{\sqcap} = \varepsilon_{ab}$, $\overset{a\,b}{\sqcup} = \varepsilon^{ab}$ denote spinor epsilons so that $\varepsilon_{12} = 1$, $\varepsilon_{21} = -1$, $\varepsilon_{11} = \varepsilon_{22} = 0$ (two indices) and $\varepsilon^{ij} = \varepsilon_{ij}$.

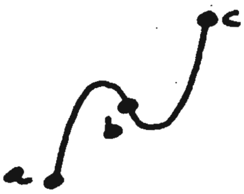
Then let $\cap = \sqrt{-1}\,\sqcap$, $\cup = \sqrt{-1}\,\sqcup$ and the Fierz identity $\asymp\; = \;)(- \times$ becomes $\asymp +)(+ \times = 0$ [under the convention that the crossing contributes a (-1) sign so that ${}^{a}_{c}\!\times^{b}_{d} = -\delta^{a}_{d}\,\delta^{b}_{c}$]. And $O = O = (\sqrt{-1})^2\, \sqcap\!\sqcup = -2$.

In matrix terms, $\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $\sqrt{-1}\,\varepsilon = \begin{bmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}$. <u>The loop value O is the sum of the squares of the entries of this matrix.</u>

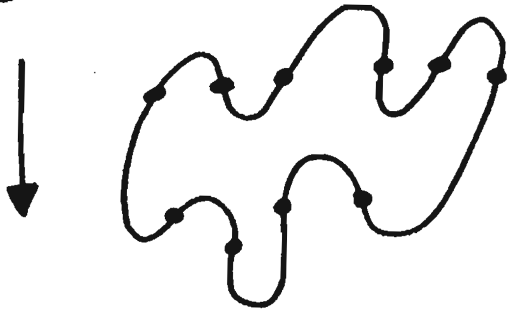
In order to create a similar model for the bracket, we need to deform $\sqrt{-1}\,\varepsilon$ so that the sum of the squares of its entries is $d = -A^2 - A^{-2}$. Therefore, let $\tilde{\varepsilon} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix}$ and let $M = \begin{bmatrix} 0 & \sqrt{-1}\,A \\ -\sqrt{-1}\,A^{-1} & 0 \end{bmatrix}$.

Let  $= M_{ab}$ and  $= M^{ab} = M_{ab}$.

Then, we have $M_{\cdot\cdot}M^{\cdot\cdot} = $]] , i.e. $M^2 = I$, and

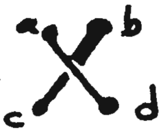
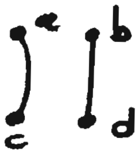
 $= M_{ab}M^{bc} = \delta_a^c =$  .

Thus any version of the loop will give $d = -A^2 - A^{-2}$



,

and one can think of $\langle K \rangle$ as a vacuum-vacuum expectation [7] with creations , annihilations  and interactions , . Bracket tells the exact form of the braiding matrices:

$R^{ab}_{cd} =$  $= A$  $+ A^{-1}$ 

and these can be seen directly to satisfy the Yang-Baxter equation [1] (corresponding to the III-move.)

The Quantum Group

$$SL(2) = \{ \mathcal{U} \mid \mathcal{U} \varepsilon \mathcal{U}^T = \varepsilon \}.$$

Therefore consider $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with associative, possibly non-commutative entries, and <u>ask</u> for

$$\boxed{\begin{array}{l} P \hat{\tilde{\varepsilon}} P^T = \hat{\tilde{\varepsilon}} \\ P^T \hat{\tilde{\varepsilon}} P = \hat{\tilde{\varepsilon}} \end{array}}.$$

Check that this is equivalent to the equations:

$$\boxed{\begin{array}{ll} ca = q\,ac & db = q\,bd \\ ba = q\,ab & dc = q\,cd \\ bc = cb & \\ ad - da = (q^{-1} - q)\,bc & \\ ad - q^{-1}bc = 1 & \end{array}} \quad \underline{q = \sqrt{A}}.$$

This algebra is the dual $\mathcal{U}^*$ of the quantum universal enveloping algebra $\mathcal{U}_q sl_2$ [1]. Thus the "quantum group" $\mathcal{U}^* = SL(2)_q$ appears naturally as the algebra of symmetries of abstract topological tensor diagrams generalizing the binor calculus.

This heralds a corresponding generalization of spin networks, and perhaps the Spin Geometry Theorem [5],[6].

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