Light Rays Near i^0 : A New Mass-Positivity Theorem

As was emphasized by Helmut Friedrich in a recent survey talk [given at the E.T. Newman birthday conference, Pittsburgh, May 1990], one of the major problems obstructing a proper understanding of asymptotically flat space-times is the lack of a completely satisfactory geometrical framework for analysing i^0 (the conformally singular point at spacelike infinity). Perhaps twistor theory can make a significant contribution here. Whereas partial understandings have come through the work of Ashtekar-Hansen, Geroch, Beig, Sachs, Sommers and many others, there remains a distinct conceptual awkwardness, and as yet there is no elegant geometrical description to match — and to unite with — that of $\mathscr{I}^{\pm}$. The "spi" construction of Ashtekar-Hansen seems to come closest [J. Math. Phys. **19** (1978) 1542] and there are indeed some relations to twistor theory. In Minkowski space $\mathbb{M}$, the points of spi can be identified with timelike hyperplanes (i.e. timelike 3-quadrics through i) and these are represented by (proportionality classes of) skew twistors $S^{\alpha\beta}$ subject to $S^{\alpha\beta}I_{\alpha\beta}=0$ and the "reality condition" $\bar{S}_{\alpha\beta}I^{\gamma\beta}=I_{\alpha\beta}S^{\gamma\beta}$. Such objects find their place in the <u>moment sequence</u> (see Spinors & Space-Time, vol. 2, pp 94, 96). The points of $\mathscr{I}$ arise when $S^{\alpha\beta}$ is <u>simple</u> (and so has the form $Z^{[\alpha}I^{\beta]\gamma}W_{\gamma}$).

I shall postpone, until a later date, a detailed discussion of the geometry involved here. In any case, we cannot expect to capture the structure of i^0 that incorporates the total mass-momentum and angular momentum of the system using merely Minkowski-space twistors. Possible schemes for involving more general types of twistor would be to use 2-surface twistors in the manner of Shaw, or perhaps to use hypersurface twistors, e.g. for a hyperboloidal timelike hypersurface that approximates $i^0 \cup \mathscr{I}^+ \cup \mathscr{I}^-$.

Of more immediate physical relevance would be to study the behaviour of light rays near i^0. There is a clear connection with twistor theory here, but for the moment I shall just show how this study can be used to give a new and perhaps comparatively simple proof of total energy positivity in general relativity (compare Schoen & Yau, Phys. Rev. Lett. **43** (1979) 1457, Commun. Math. Phys **65** (1979) 45; Witten, Commun. Math. Phys **80** (1981) 381):

<u>Theorem</u> If $\mathcal{M}$ is asymptotically simple and satisfies the strong null convergence condition, then it cannot have a well-defined negative mass at i^0.

<u>N.B.</u> By the "strong" <u>null convergence condition</u> (**SNCC**) I shall mean that every endless null geodesic contains a pair of conjugate points. This is a consequence of the three properties:

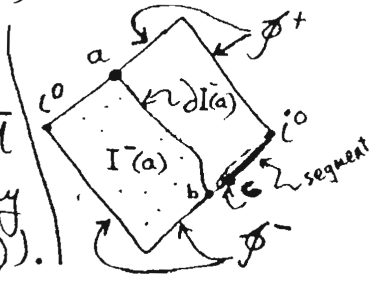
(1) $R_{ab}n^a n^b \leq 0$ for all null vectors n^a (i.e. with Einstein's equations, & my sign conventions, $T_{ab}n^a n^b \geq 0$),

(2) null geodesic completeness, and

(3) the genericity condition: $k_{[e}R_{a]bc[d}k_{f]}n^b n^c \neq 0$ somewhere along each null geodesic. [see Hawking & Penrose, Proc. Roy. Soc. (Lond) A314 (1970) 529.]

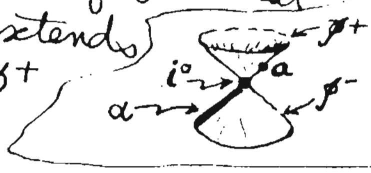
<u>Outline of proof</u>

The idea is to assume that the mass at i^0 <u>is</u> negative and that SNCC holds, and then to derive a contradiction. Let $a \in \mathscr{I}^+$ and consider $I^-(a)$. (All sets are in $\bar{\mathcal{M}} = \mathcal{M} \cup \mathscr{I}^+ \cup \mathscr{I}^-$; here "$I^-(a)$" is to include its limit points (interior limit points only, so that $I^-(a)$ remains open) on $\mathscr{I}^-$; "$\partial I^-(a)$" is supposed to be composed of the points of $\partial I^-(a)$ in $\mathcal{M}$, together with the limit points thereof in $\bar{\mathcal{M}}$.) Let us see what would happen if it could be shown that some segment of a generator of $\mathscr{I}^-$, from a point $c \in \mathscr{I}^-$ to i^0, has a neighbourhood in $\bar{\mathcal{M}}$ that does not meet $I^-(a)$ (— as follows if any $c \in \mathscr{I}^-$ has a neighbourhood not meeting $I^-(a)$).

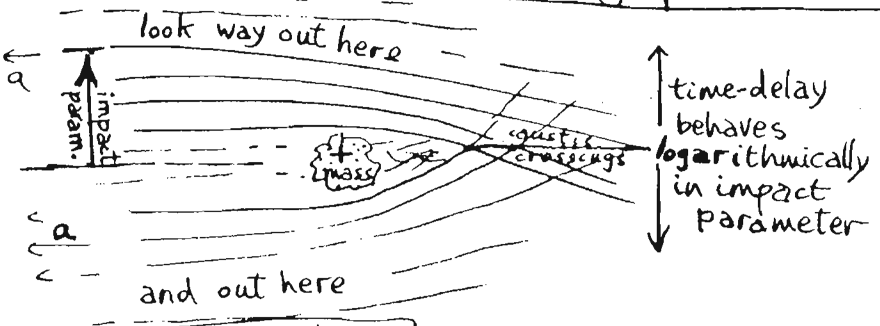


In that case $\partial I^-(a)$ would meet $\mathscr{I}^-$ transversely in some (non-vacuous) set $\mathscr{B}$ $(= \partial I^-(a) \cap \mathscr{I}^-)$. Let $b \in \mathscr{B}$. By standard theorems (since $\mathcal{M}$ is globally hyperbolic, as follows from its asymptotic simplicity) there must be a null geodesic from **b** to **a**, lying on $\partial I^-(a)$, with no pair of conjugate points between b and a, which contradicts SNCC.

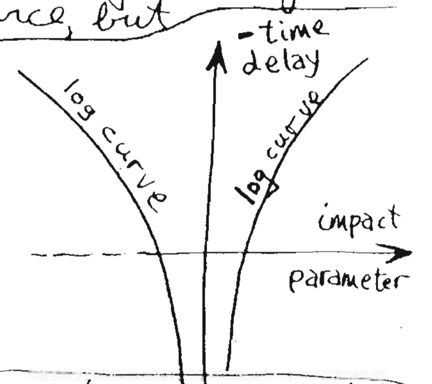
In fact there is <u>no</u> such segment (or point c), but what <u>does</u> happen, with negative mass at i^0, is almost as effective. Let us examine the generators of the past light cone $\mathscr{A}$ of **a** (we have $\partial I^-(a) \subset \mathscr{A}$), particularly in the neighbourhood of that generator α of $\mathscr{I}^-$ that is "diametrically opposite" to **a** (i.e. α extends through i^0 to become the generator of $\mathscr{I}^+$ containing **a**). We shall be mainly concerned with generators of $\mathscr{A}$ "close" to that particular generator of $\mathscr{I}^+$ on which **a** lies (i.e. α extended), and with their intersections with $\mathscr{I}^-$.



To assist us in picturing this situation, consider first the case of <u>positive</u> mass at i^0, to see why there is no conflict with SNCC. Not only are the light rays deflected inwards as they pass the source, but



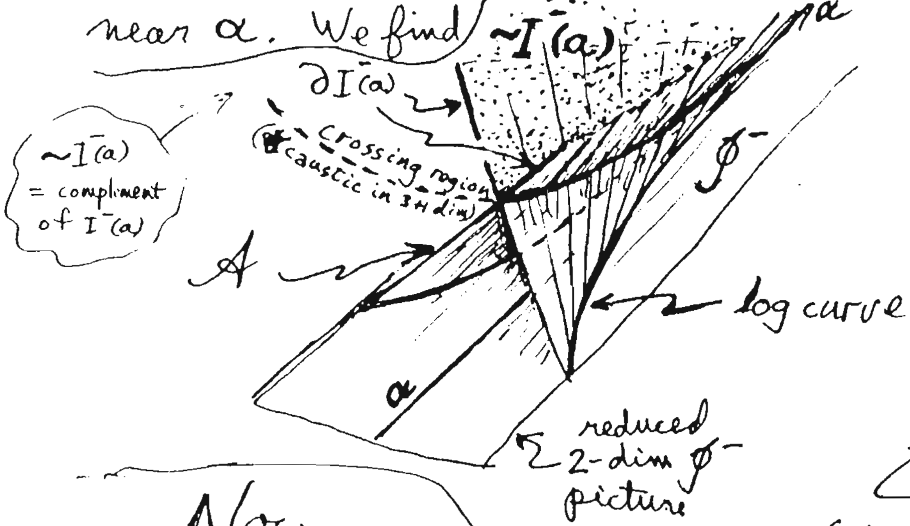
Spatial picture

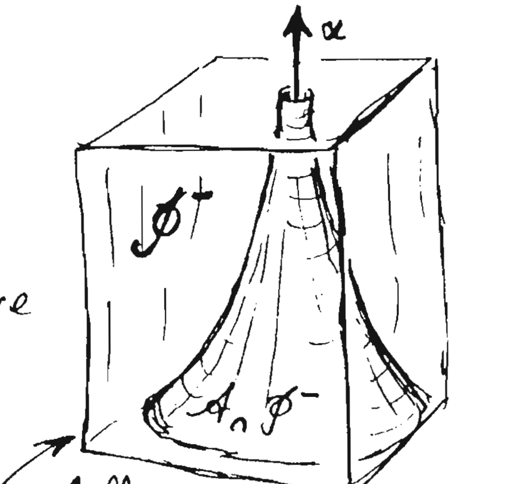


Graph of time-delay

there is also a time-delay that behaves logarithmically in the impact parameter (distance of "closest approach"). This has no natural zero, so the larger the impact param., the larger the values of advanced time that will eventually be encountered. This phenomenon was described in detail by R.P. in the Taub Festschrift volume [ed. F. Tipler, ~1980]. We find that the whole of $\mathscr{I}^-$ is contained in $I^-(a)$. (Hence $\partial I^-(a) \cap \mathscr{I}^- = \emptyset$, so no contradiction with SNCC.)

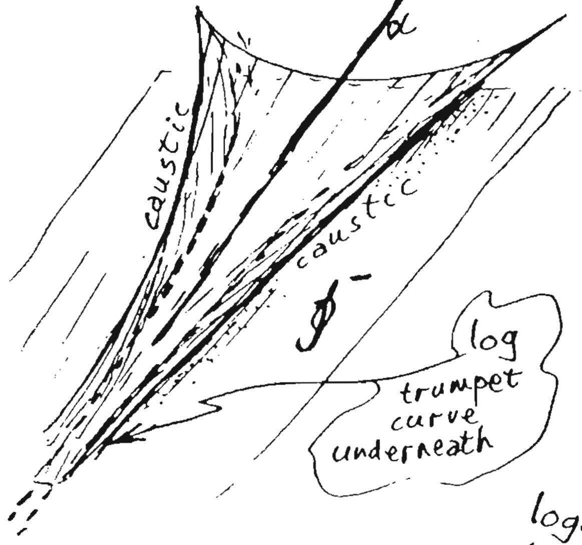
To see how this comes about, examine $\mathcal{A}$ and $\partial I^-(a)$ near α. We find

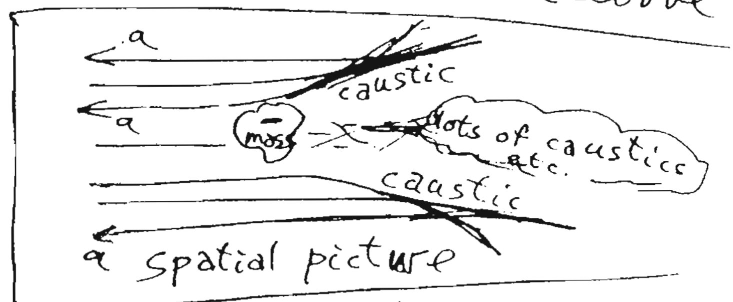


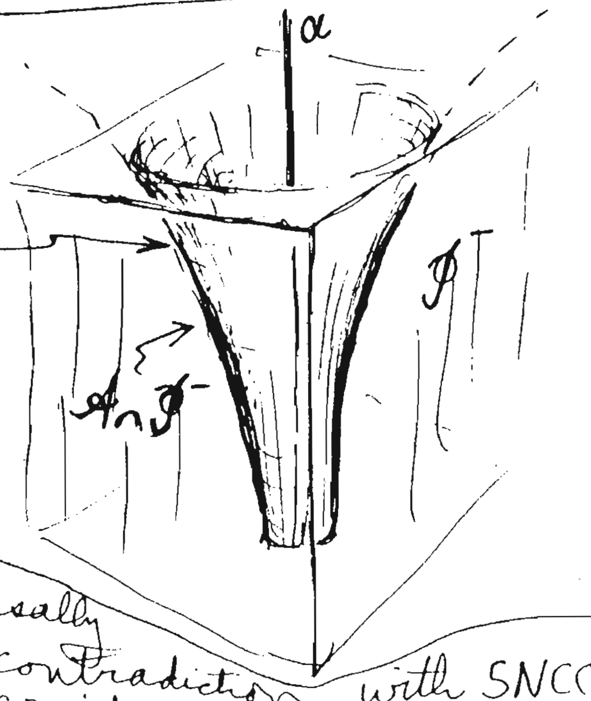


full 3-dim $\mathscr{I}^-$ picture ($\mathcal{A} \cap \mathscr{I}^-$ is logarithmic trumpet)

Now consider negative mass. The picture is like the one above but the other way up:



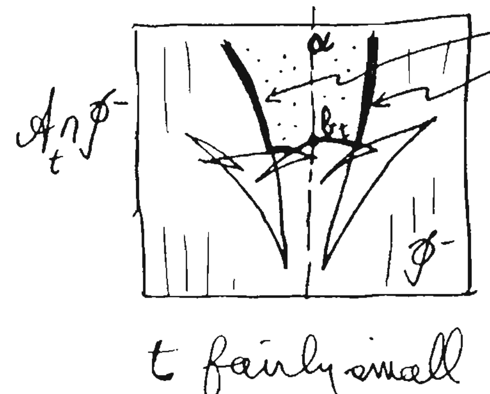




What's not so clear from this picture is the fact that $\mathscr{I}^- \subset \overline{I^-(a)}$ $(= I^-(a) \cup \partial I^-(a))$ so that, as in the positive and zero mass cases $\partial I^-(a)$ does not intersect $\mathscr{I}^-$ transversally and we have no immediate contradiction with SNCC.

To see what does happen, consider a smooth family of points a_t $(t \geq 0)$ lying on (say) a null ray terminating at α, where $a = a_0$. For each $t > 0$, $\partial I^-(a)$ intersects $\mathscr{I}$

$\mathscr{I}^-$ transversally in a (not necessarily smooth) cut of $\mathscr{I}^-$. The point $b_t = \alpha \cap \partial I^-(a_t)$ lies on a null ray on $\partial I^-(a_t)$ free of conjugate points (except possibly at end-points). As $t \to 0$, the intersection $\mathcal{A}_t \cap \mathscr{I}^-$ (where $\mathcal{A}_t$ is past light cone of a_t) looks like:



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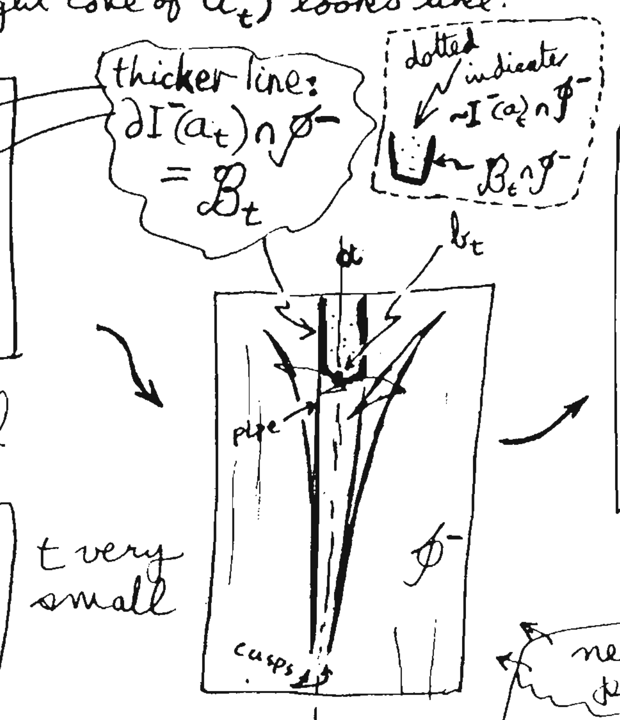
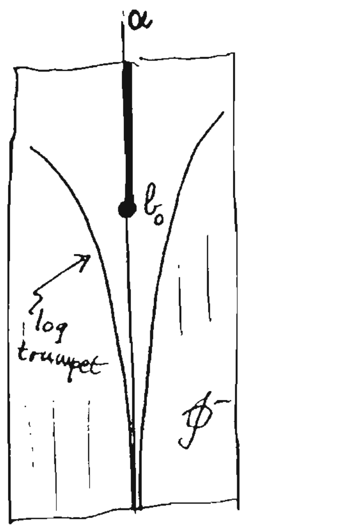


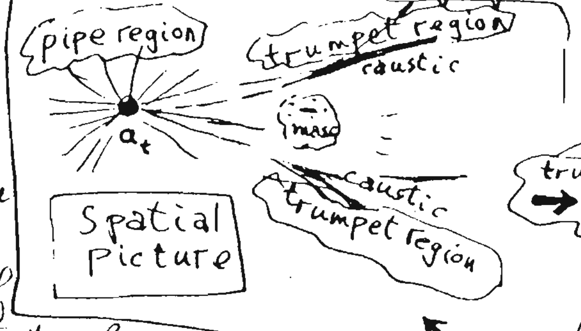


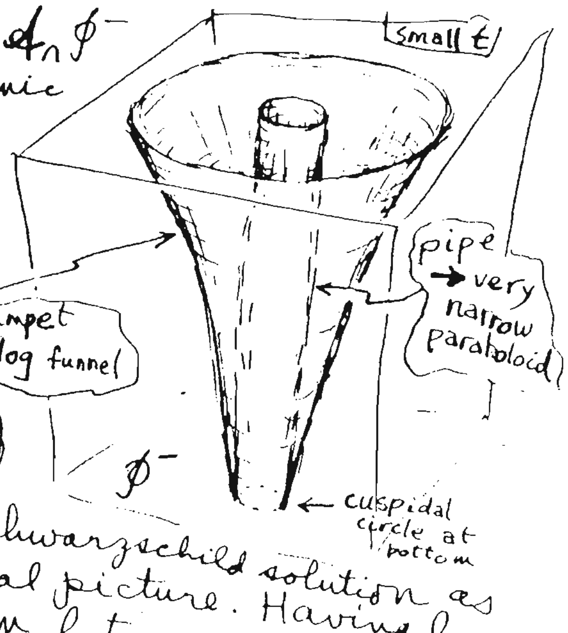
$t = 0$

negative mass pictures

The region $\mathcal{B}_t = \partial I^-(a_t) \cap \mathscr{I}^-$ moves continuously with t, finally converging on a future-endless segment of α with past end-point $b_0 = b$.

One must show that the part of $\mathcal{A}_t \cap \mathscr{I}^-$ that lies away from the logarithmic trumpet and the narrower pipe within it actually stabilizes as $t \to 0$ so that b_t attains a limit. The pipe and trumpet come from the resemblance of $\mathcal{M}$'s i^0 to that of a negative mass Schwarzschild solution as can be seen from examining the spatial picture. Having b_0, we find the null ray on $\partial I^-(a_0)$ from b_0 to a_0, which must be free of conjugate points: the desired contradiction with SNCC.





No doubt the argument can be strengthened in various ways (e.g. allowing conj. pts at $\mathscr{I}^\pm$ or the mass to be zero). Further details elsewhere.

~ Roger Penrose