A quasi-local mass construction with positive mass

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In this note we propose a pair of modifications to Penrose's quasi-local mass construction that not only lead to a definition of a real 4-momentum and mass of the gravitational and matter fields within a two surface $\mathcal{S}$, but also have the property that the momentum can be proved to be future pointing when the 2-surface can be spanned by a three surface on which the data satisfies the dominant energy condition (the proof also requires that the 2-surface be convex). The new definition reproduces the good properties of the quasi-local mass construction—it gives zero in flat space, and the correct results in linearized theory and at infinity.

Motivation: In Mason (1989) (see also Mason & Frauendiener 1990) the components of the angular momentum twistor associated to a 2-surface $\mathcal{S}$ are interpreted as the values of the Hamiltonians that generate motions of a spanning 3-surface $\mathcal{H}$ whose boundary value on $\mathcal{S}$ are 'quasi-Killing vectors' constructed out of solutions to the 2-surface twistor equations. (The value of the Hamiltonians that generate motions of $\mathcal{H}$ in space-time depends only on the boundary value of the deformation 4-vector field on $\mathcal{S}$.)

For Penrose's quasi-local mass construction (Penrose 1982) the quasi-Killing vectors are constructed out of the four linearly independent solutions of the twistor equation $\omega^A_\alpha = (\omega^A_0, \ldots, \omega^A_3)$. They are given by $K^{AA'} = K^{\alpha\beta}\omega^A_\alpha \pi^{A'}_\beta$ where $K^{\alpha\beta} = K^{(\alpha\beta)}$ is a matrix of constants and $\pi_{A'\alpha}$ are the π-parts of the ω^A_α defined by $d\omega^A|_{\mathcal{S}} = -i\pi_{A'}dx^{AA'}|_{\mathcal{S}}$. The value of the Hamiltonian that generates deformations of $\mathcal{H}$ with boundary value $K^{AA'}$ on $\mathcal{S}$ is obtained by inserting this decomposition of $K^{AA'}$ into the Witten-Nester integral:

$$H(K^{AA'}) = A_{\alpha\beta}K^{\alpha\beta} = -i\oint_{\mathcal{S}} K^{\alpha\beta}\omega^A_\alpha d\pi^{A'}_\beta \wedge dx_{AA'}.$$

This expression depends on $K^{AA'}$ *and* its decomposition into spinors. By use of $d(-i\pi_{A'}dx^{AA'}) = d^2\omega^A = R_B{}^A\omega^B$ it can be seen that this is equivalent to Penrose's original definition.

The new momentum definition: In order to define a real 4-momentum we must have a definition of real 'quasi-translations' at $\mathcal{S}$. Two definitions follow. The equation $\bar{\eth}\pi_{A'} = 0$, resp. $\eth\pi_{A'} = 0$ (where $\bar{\eth} = \bar{m}^a\nabla_a$, $\eth = m^a\nabla_a$, $\bar{m}^a = \iota^A o^{A'}$ and $m^a = o^A\iota^{A'}$ with $o^{A'}o^A$, $\iota^A\iota^{A'}$ the outward resp. inward null normal etc.) in general has just 2 linearly independent solutions on $\mathcal{S}$, since this equation can be thought of as the condition that $\pi_{A'}$ is a holomorphic (resp. anti-holomorphic) section of the spin bundle $\mathbf{S}_{A'}$ on the sphere $\mathcal{S}$ where the complex structure on $\mathcal{S}$ is that induced from the space-time metric, and that on $\mathbf{S}_{A'}$ arises from the space-time spin connection. Generically $\mathbf{S}_{A'}$ is trivial as a holomorphic vector bundle on $\mathcal{S}$ and so there exists precisely two solutions $(\pi^{0'}_{A'}, \pi^{1'}_{A'}) = \pi^{A'}_{A'}$. (This type of idea is used in KPT's 1983 definition of quasi-local charges for Yang-Mills.)

We can now define a 'quasi-translation' to be a 4-vector field on $\mathcal{S}$ of the form

$$K_{AA'} = K_{\underline{A}\underline{A}'}\pi^{\underline{A}'}_{A'}\bar{\pi}^{\underline{A}}_{A}$$

where the $K_{\underline{A}\underline{A}'}$ are constants. This can now be inserted into the Witten-Nester form to obtain the corresponding values of the momenta. The quasi-local momentum can thus be defined (modulo irrelevant overall real constants) as:

$$P^{\underline{A}\underline{A}'} = i\oint_{\mathcal{S}} \pi^{\underline{A}'}_{A'} d\bar{\pi}^{\underline{A}}_{A} \wedge dx^{AA'}$$

The mass. In order to define a mass, we must be able to define a constant $\varepsilon_{\underline{A}\underline{B}}$ so that we can define:

$$m^2 = P^{\underline{A}\underline{A}'}P^{\underline{B}\underline{B}'}\varepsilon_{\underline{A}\underline{B}}\varepsilon_{\underline{A}'\underline{B}'}.$$

The natural definition is $\varepsilon^{\underline{A}'\underline{B}'} = \pi^{\underline{A}'}_{A'}\pi^{\underline{B}'}_{B'}\varepsilon^{A'B'}$. It follows from $\bar{\eth}\pi^{\underline{A}'}_{A'} = 0$ that $\bar{\eth}\varepsilon^{\underline{A}'\underline{B}'} = 0$, so that the $\varepsilon^{\underline{A}'\underline{B}'}$ are holomorphic and global functions on the sphere and hence, by Liouville's theorem, constant.

Flat space, linearised theory and infinity. In flat space, the $\pi_{A'}$ are guaranteed to be the restriction to $\mathcal{S}$ of the constant spinors, since they certainly satisfy the equation, and the solutions are unique. The integrand therefore vanishes giving the correct answer $P_{\underline{A}\underline{A}'} = 0$. In linearized theory one can again, with a little work, see that the right answer is obtained (one needs to integrate potentially awkward terms by parts in order to see that they vanish). Asymptotically at space-like infinity, the $\pi_{A'}$'s are the asymptotically constant spinors (again because the asymptotically constant spinors satisfy $\bar{\eth}\pi_{A'} = 0$ and therefore span the solution space) and the expression reduces to the Witten-Nester expression for the ADM energy. At null infinity there is the subtlety that only one of the definitions gives the correct asymptotic spin space depending on whether one is at future or past null infinity.

Positivity. It is essential for a good definition of momentum that it should be future pointing. The following argument is an adaptation of ideas in Ludvigsen & Vickers (1983) based on Witten (1981). In the following we show that $P^{00'}$ is positive, and write, for simplicity, $\pi_{A'} = \pi^{0'}_{A'}$.

> ***Theorem.*** The quasi-local momentum $P^{00'}$ defined by the $\eth\pi_{A'} = 0$ (resp. $\bar{\eth}\pi_{A'} = 0$) is positive whenever $\rho < 0$ (resp. $\rho' > 0$).

Proof. Let $\lambda_{A'}$ be some field defined on a 2-surface $\mathcal{S}$ spanned by some non singular space-like 3-surface $\mathcal{H}$. Let $I_\lambda(\mathcal{S})$ be the integral of the Witten-Nester 2-form $\Lambda = -i\bar{\lambda}_A d\lambda_{A'} \wedge dx^{AA'}$ over $\mathcal{S}$. In spin coefficients and the GHP formalism this may be written:

$$I_\lambda(\mathcal{S}) = \oint_{\mathcal{S}} \{\bar{\lambda}_1(\eth\lambda_{0'} + \rho\lambda_{1'}) - \bar{\lambda}_0(\bar{\eth}\lambda_{1'} + \rho'\lambda_{0'})\}dS \qquad (1).$$

Consider first the system of equations $\eth\pi_{A'} = 0$:

$$\eth\pi_{0'} + \rho\pi_{1'} = 0, \qquad \eth\pi_{1'} + \bar{\sigma}'\pi_{0'} = 0 \qquad (2,3).$$

Then using (2) and integrating by parts we get:

$$I_\pi(\mathcal{S}) = -\oint_{\mathcal{S}} (\rho'\bar{\pi}_0\pi_{0'} + \rho\bar{\pi}_1\pi_{1'})dS \qquad (4)$$

Since the Sen-Witten equation on $\mathcal{H}$ consists of an elliptic system of two first order P.D.E's, we may find a solution $\tilde{\pi}_{A'}$ satisfying the boundary condition

$$\tilde{\pi}_{0'} = \pi_{0'} \qquad (5)$$

on $\mathcal{S}$. In general $\tilde{\pi}_{1'}$ will differ from $\pi_{1'}$ on $\mathcal{S}$. Denote this difference by:

$$Y = \tilde{\pi}_{1'} - \pi_{1'} \qquad (6)$$

We now relate $I_\pi(\mathcal{S})$ to $I_{\tilde{\pi}}(\mathcal{S})$:

$$\begin{aligned}
I_{\tilde{\pi}}(\mathcal{S}) &= \oint_{\mathcal{S}} \left\{\bar{\tilde{\pi}}_1(\eth\tilde{\pi}_{0'} + \rho\tilde{\pi}_{1'}) - \bar{\tilde{\pi}}_0(\bar{\eth}\tilde{\pi}_{1'} + \rho'\tilde{\pi}_{0'})\right\}dS \\
&= \oint_{\mathcal{S}} \left\{\bar{\tilde{\pi}}_1(\eth\pi_{0'} + \rho\tilde{\pi}_{1'}) - \bar{\pi}_0(\bar{\eth}\tilde{\pi}_{1'} + \rho'\pi_{0'})\right\}dS \\
&= \oint_{\mathcal{S}} \left\{\bar{\tilde{\pi}}_1(-\rho\pi_{1'} + \rho\tilde{\pi}_{1'}) - \rho\bar{\pi}_1\tilde{\pi}_{1'} - \rho'\bar{\pi}_0\pi_{0'})\right\}dS \\
&= \oint_{\mathcal{S}} \left\{-\rho'\bar{\pi}_0\pi_{0'} - \rho\bar{\pi}_1\pi_{1'} + \rho(\bar{\tilde{\pi}}_1 - \bar{\pi}_1)(\tilde{\pi}_{1'} - \pi_{1'})\right\}dS \\
&= I_\pi(\mathcal{S}) + \oint_{\mathcal{S}} \rho Y\bar{Y}dS
\end{aligned}$$

Where we have used equations (2), (4), (5) and (6) and an integration by parts. As is well known (Witten 1980) for matter satisfying the dominant energy condition $I_{\tilde{\pi}}(\mathcal{S}) \geq 0$ so that whenever $\rho \leq 0$, $I_\pi(\mathcal{S}) \geq 0$. This implies that $P^{\underline{A}\underline{A}'}$ is future pointing as required.

Considering next the equation $\bar{\eth}\pi_{A'} = 0$, an analogous argument to the one above but now with $\tilde{\pi}_{1'} = \pi_{1'}$ as boundary conditions for the Sen-Witten equation will show positivity whenever $\rho' \geq 0$.□

The conditions $\rho \leq 0$ or $\rho' \geq 0$ are the condition that the two surface is convex, i.e. that there should be no indentations. This will be satisfied by a wide class of 2-surfaces in a generic space-time.

Angular momentum: One can define more general quasi-Killing vectors using local twistors, $(\omega^A, \pi_{A'})$ restricted to $\mathcal{S}$ satifying either $\eth(\omega^A, \pi_{A'}) = 0$ or $\bar{\eth}(\omega^A, \pi_{A'}) = 0$ where $\eth$ and $\bar{\eth}$ act according to the local twistor connection. These equations are guaranteed to have just four independent solutions generically since as before these are $\bar{\partial}$ equations whose solutions are holomorphic sections of a holomorphic vector bundle on the sphere $\mathcal{S}$. Generically the holomorphic vector bundle will be trivial and so there will be just four linearly independent solutions. These can be used to define quasi-Killing vectors, and quasi-conformal Killing vectors as in Mason & Frauendiener which then give rise to 'conserved' quantities by substitution into the Witten-Nester form. (When $R_{ab} = 0$ on $\mathcal{S}$, it makes consistent sense to set $\pi_{A'} = 0$ in such a local twistor and then we can retrieve the quasi-local momentum above within the scheme.)

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