Monopoles and Yang-Baxter equations

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It has been known for some time that the Yang-Baxter equations can be solved using elliptic curves. More recently it was discovered [1, 2] that the YBE for the N state chiral Potts model could be solved using special curves of genus $(N-1)^2$.

The curves that arise can be defined as follows. Let $k^2 + k'^2 = 1$ and then intersect the following two Fermat surfaces in $\mathbb{CP}_3$:

$$a^N + k' b^N = k d^N$$
$$k' a^N + b^N = k c^N.$$

This gives a curve B_N with a high degree of symmetry. In fact $\mathbb{Z}_N^4$ acts on $\mathbb{CP}_3$ by $(\omega_1, \omega_2, \omega_3, \omega_4)[a, b, c, d] = [\omega_1 a, \omega_2 b, \omega_3 c, \omega_4 d]$ (where $\omega_i^N = 1$) and this fixes B_N. (Of course the diagonal $(\omega, \omega, \omega, \omega)$ acts trivially so it is really a $\mathbb{Z}_N^3 = \mathbb{Z}_N^4/\Delta$ action). The quotient of B_N by a free action of a $\mathbb{Z}_N$ subgroup gives the curve Σ_N of genus $(N-1)^2$.

On a visit to Canberra in 1989 the first author conjectured that these curves should be related to the spectral curves of an $SU(2)$ monopole of charge N. These are also a special class of curves of genus $(N-1)^2$. We now understand how such a relationship exists for hyperbolic monopoles with Higgs field equal to zero. Consider the $\mathbb{C}^\times$ action on $\mathbb{CP}_3$ given by

$\lambda\,[a,b,c,d] \longmapsto [\lambda a, b, c, \lambda d]$. If we remove the lines $C_1 = [0,b,c,0]$ and $C_2 = [a,0,0,d]$ of fixed points this is a free action with quotient the quadric $Q = \mathbb{P}_1 \times \mathbb{P}_1$. In fact it realizes $\mathbb{CP}_3 - C_1 \cup C_2$ as the $\mathbb{C}^\times$ bundle of the line bundle $\mathcal{O}(1,-1)$ over the quadric. We shall call this bundle L.

The important fact is that B_N intersects the orbits of the $\mathbb{C}^\times$ action in the orbits of $\mathbb{Z}_N$ considered as a subgroup inside $\mathbb{C}^\times$. So projecting to Q divides B_N by $\mathbb{Z}_N$ to give the curve Σ_N in Q. It is easy to check that Σ_N is in the linear system $\mathcal{O}(N,N)$. In fact we can say more. If we factor the $\mathbb{C}^\times$ bundle $\mathbb{C}P_3 - C_1 \cup C_2$ by $\mathbb{Z}_N$ this gives the bundle $L^N \to Q$ and the curve B_N becomes a section over Σ_N. So Σ_N satisfies the constraint

$$L^N/_{\Sigma_N} \simeq \mathcal{O}.$$

In the theory of hyperbolic monopoles [3] the monopole is determined by a spectral curve S_N in Q in the linear system $\mathcal{O}(N,N)$. This satisfies a constraint

$$L^{2p+N}/_{S_N} \simeq \mathcal{O}.$$

where p is the norm of the Higgs field at infinity.

So this shows that the curve Σ_N is that for a monopole with zero Higgs field! Strictly speaking such monopoles are trivial so we have to interpret Σ_N as arising from some limit of monopoles. Work in progress suggests that this can be done via the rational map of the monopole.

Finally notice that we can turn this discussion about and say that a curve Σ_N as above with the constraint $L^N/_{\Sigma_N} \simeq \mathcal{O}$ is equivalent to a curve in $\mathbb{CP}_3$ with no constraint except invariance under $\mathbb{Z}_N \subset \mathbb{C}^\times$. Looked at from this point of view Baxters curves B_N are special curves invariant under two more actions of $\mathbb{Z}_N$. The more general curves we have discussed here have moduli spaces of dimension $4N$ and it is hoped that this means that Baxter's curves can be generalised.

[1] Commuting transfer matrices in the Chiral Potts Models: Solutions of Star-triangle equations with genus > 1 Phys. Lett. A. 123 5, 219–223, 1987
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[2] New solutions of the Star-triangle relations for the Chiral Potts model.
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[3] Magnetic monopoles in hyperbolic spaces.
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