H-Space—a universal integrable system?

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The following speculations have not been fulfilled yet (and may never) but I feel that the concrete aspects of the ideas are of interest and the various relations involved are intriguing.

Motivation. There is a large forest of integrable systems. Richard Ward, amongst others, has pointed out that many, if not indeed most integrable systems are reductions of the self-dual Yang-Mills equations. This observation isn't just a question of bookkeeping, it gives a substantial insight into the theory underlying these equations as the inverse scattering transform for these systems can be understood as a symmetry reduction of the Ward construction for solutions of the self-dual Yang-Mills equations (Woodhouse & Mason 1988 and Mason & Sparling 1989 & preprint, the symmetry reduction can, however, be somewhat nontrivial—see in particular Woodhouse & Mason in which non-Hausdorff Riemann surfaces play an essential role).

Two gaps in the story are as follows. Firstly that there appears to be genuine difficulties to incorporating the KP and Davey-Stewartson equations. There is little difficulty in incorporating integrable systems into some kind of twistor framework if the inverse scattering transform is realised by means of the solution of a Riemann-Hilbert problem. However the inverse scattering problem for the KP equations is more subtle and requires the solution of a 'non-local Riemann-Hilbert problem'. This gap is particularly irritating in view of the theoretical importance that the KP equations have acquired with its relations to the theory of Riemann surfaces etc. The second gap is that there appears to be little rôle for the self-dual vacuum equations and its twistor construction, RP's nonlinear graviton construction—this, it should be pointed out, is not based on the solution of a Riemann-Hilbert problem either. However I should like to make the following conjecture:

Conjecture. The KP and Davey-Stewartson equations are reductions of the self-dual Einstein equations.

The circumstantial evidence is as follows. (The self-duality equations are taken to be concerned with space-times with metric of signature (2,2).)

Lemma 1. KP can be obtained in the limit as $n\to\infty$ of the $SL(n)$ self-dual Yang-Mills equations reduced by two orthogonal null translations.
(This extends the results of Mason & Sparling 1989.)

Lemma 2. (Hoppe, J.) The Lie algebra of the area preserving diffeomorphism group of a surface Σ^2, $SDiff(\Sigma^2)$ can be approximated arbitrarily closely by that of $SL(n)$ as $n\to\infty$.

Lemma 3. The self-dual Einstein equations are equivalent to the self-dual Yang-Mills equations reduced by two orthogonal null translations with gauge group $SDiff(\Sigma^2)$. (This extends the results of Mason & Newman 1989)□

Remark. If it were the case that $SL(n)$ were a subgroup of $SL(\infty) = SDiff(\Sigma^2)$ then these results would imply that all 2-dimensional integrable models obtainable as reductions from the self-dual Yang-Mills equations (at least by translations). Hence the title of this note and the question mark. However, my current opinion is that $SL(n)$ is only a subgroup of $SDiff(\Sigma^2)$ for $n = 2$. This still yields a reasonable class of integrable systems and certainly the more famous ones such as the KdV, nonlinear Schrodinger and the sine-Gordon equations.

Proof of lemma 1. I shall use the presentation of the KP hierarchy due to Gelfand and Dicke. See for instance Segal & Wilson in the proceedings of the I.H.E.S for a description of these ideas. The equations of the KP hierarchy are the consistency conditions for the existence of a solution ψ to the following system of linear partial differential equations

$$(\partial_{t_2} - (Q^2)_+)\psi = 0,\ (\partial_{t_3} - (Q^3)_+)\psi = 0,\ \cdots,\ (\partial_{t_r} - (Q^r)_+)\psi = 0,\ \cdots$$

where $(Q^r)_+$ is an r^{th} order O.D.E. in the x variable, $Q_r = (\partial_x)^r + ru(\partial_x)^{r-2} + \cdots + w_r$ and $u(x, t_2, t_3, \cdots)$ is the subject of the KP hierarchy equation and w_r is some function which will be determined in terms of u by the equations. The notation is intended to indicate that the ordinary differential operators $(Q^r)_+$ are the differential operator part of the pseudo-differential operator Q raised to the r^{th} power where $Q = \partial_x + u(\partial_x)^{-1} +$ (lower order) and where $(\partial_x)^{-1}$ is a formal pseudo-differential operator defined by the relation $(\partial_x)^{-1}f = f(\partial_x)^{-1} + \sum_{i=1}^{\infty}(-\partial_x)^i f(\partial_x)^{-i-1}$.

The basic KP equation is the equation on $u(x, t_1, t_2)$ that follows from the consistency conditions for $(\partial_{t_2} - (Q^2)_+)\psi = 0$ and $(\partial_{t_3} - (Q^3)_+)\psi = 0$ alone. The evolution in the higher time variables are symmetries of the basic equations (and each other). If one imposes invariance in the n^{th} time variable t_n, then the reduced system is referred to as the n^{th} generalized KdV hierarchy ($n = 2$ gives the standard KdV hierarchy and $n = 3$ the Boussinesq).

The basic idea is that the operators $(Q^r)_+$ can be thought of as infinite dimensional matrices acting on $L^2(\mathbf{R})$ where x is a coordinate on $\mathbf{R}$. One can approximate this by $n \times n$ matrices by imposing a symmetry in the n^{th} time variable since then (Fourier transforming ψ in the t_n variable) we have $(Q^n)_+\psi = \lambda\psi$ and we consider only ψ lying in the n-dimensional solution space of this equation, represented, say, by ψ and its first $(n-1)$-derivatives with respect to x. With this reduction we have:

$$(\partial_{t_2} - (Q^2)_+)\psi = 0 \text{ reduces to } \left\{\partial_{t_2} - \begin{pmatrix} 2u & 0 & 1 \cdots 0_{n-3} \\ \cdot & 2u & \ddots \quad \vdots \\ \cdot & \cdot & \ddots \quad 1 \\ \cdot & \cdot & \ddots \quad 0 \\ \cdot & \cdot & \cdot\ (2-n)u \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 1 & 0 & \ddots & 0 \\ 0 & 1 & \cdots & 0 \end{pmatrix}\right\}\underline{\psi} = 0$$

$(\partial_{t_3} - (Q^3)_+)\psi = 0$ reduces to

$$\left\{ \partial_{t_2} - \begin{pmatrix} u_x & u & 0 & 1 & 0_{n-4} \\ \cdot & \cdot\, u_x & u & \ddots & 1 \\ \cdot & \cdot & \cdot & \ddots & 0 \\ \cdot & \cdot & \cdot & \cdot & \ddots \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & 1\cdots & 0 \end{pmatrix} \right\} \underline{\psi} = 0$$

where 0_r is the $r \times r$ zero matrix. This matrix linear system is linear in the spectral parameter λ and can be seen to be the linear system of a reduction of SDYM with 2 null orthogonal translation symmetries.□

Note: A large gap in the above discussion is that the linear system is shown to be contained within the SDYM linear systems, but I have not characterised those SDYM solutions with the 2 orthogonal null symmetries that give rise to the n^{th} KdV system.

Proof of Lemma 2. These ideas are standard. One presents the Lie algebra of the area preserving diffeomorphisms of a torus by using the area form as a symplectic form and representing vector fields corresponding to elements of $LieSDiff(\Sigma^2)$ by their Hamiltonians. Let θ_1 and θ_2 be angular coordinates on the torus such that the area form is $d\theta_1 \wedge d\theta_2$, then a basis for the Hamiltonians is $H_{\underline{A}} = exp\{2\pi i(A_1\theta_1 + A_2\theta_2)\}$ where $\underline{A} = (A_1, A_2) \in \mathbb{Z} \times \mathbb{Z}$. The Lie bracket is the Poisson bracket:

$$\{H_{\underline{A}}, H_{\underline{B}}\} = (\underline{A} \wedge \underline{B}) H_{\underline{A}+\underline{B}} \quad \text{where} \quad (\underline{A} \wedge \underline{B}) = A_1B_2 - A_2B_1.$$

For $SL(N)$ we use a basis for the Lie algebra constructed using a pair of matrices U, V satisfying the quantum plane relations: $UV = \zeta VU$ where $\zeta^N = 1$. An explicit representation has U diagonal with powers of ζ down the diagonal $U_{ij} = \zeta^i \delta_{ij}$ and V a shift matrix $V_{ij} = \delta_{i(j+1 \bmod N)}$.

A basis for the Lie algebra of $SL(N)$ is then furnished by

$$T_{\underline{A}} = N\zeta^{\frac{A_1A_2}{2}} U^{A_1} V^{A_2}.$$

The commutators are then given by:

$$[T_{\underline{A}}, T_{\underline{B}}] = N \sin\frac{2\pi \underline{A} \wedge \underline{B}}{N} T_{\underline{A}+\underline{B}} \quad \xrightarrow[N\to\infty]{} \quad (\underline{A} \wedge \underline{B}) T_{\underline{A}+\underline{B}}$$

which gives the same commutation relations as above for $H_{\underline{A}}$ in the limit as $N \to \infty$.□

Proof of lemma 3. This is, to a certain extent, a corollorary of the results in Mason & Newman (1989). In that paper it was shown that if you take the algebraic relations obtained by imposing four translational symmetries on the self-dual Yang-Mills equations and take the gauge group to be the group of volume preserving diffeomorphisms of some 4-manifold then, roughly speaking, one obtains the self-dual vacuum equations. Lemma 3 can be reformulated so as to be a special case of this.

The self-dual Yang-Mills equations on $\mathbf{R}^4$ with metric $ds^2 = dudy + dvdx$ (signature 2,2) are the integrability conditions on connection components (A_u, A_v, A_x, A_y) in the Lie algebra of the gauge group for the the linear system

$$\{\partial_u + A_u + \lambda(\partial_x + A_x)\}\psi = 0 \qquad \{\partial_v + A_v + \lambda(\partial_y + A_y)\}\psi = 0.$$

When G is $SDiff(\Sigma^2)$ the connection components are all vector fields on Σ^2 (depending also on the coordinates on $\mathbf{R}^4$). Impose two translational symmetries on the $\mathbf{R}^4$ so that the connection components depend only on the quotient variables on $\mathbf{R}^2$. The linear system then reduces to the system $\{V_u + \lambda V_x\}\psi = 0 = \{V_v + \lambda V_y\}\psi$ where the V's are vector fields on $\mathbf{R}^2 \times \Sigma^2$. These vector fields preserve the natural volume form on $\mathbf{R}^2 \times \Sigma^2$ and so determine elements of the Lie algebra of the volume preserving diffeomorphism group. The linear system is precisely that for the self-dual Yang-Mills equations with 4 translational symmetries and gauge group the volume preserving diffeomorphisms of $\mathbf{R}^2 \times \Sigma^2$.

Concretely introduce coordinates (p, q) on Σ^2 so that the area form is the symplectic form $dp \wedge dq$, and suppose the symmetries to be in the x and y directions so that the variables depend only on the coordinates (u, v) on $\mathbf{R}^2$. Represent the vector fields A_x on Σ^2 by their Hamiltonians denoted h_x etc.. The field equations are

$$[\partial_u + A_u + \lambda A_x, \partial_v + A_v + \lambda A_y] = 0$$

The first implication of this is that $\lambda^2[A_x, A_y] = 0$ so that we can choose coordinates on Σ^2 so that $A_x = \partial_q$ and $A_y = \partial_p$. The term proportional to λ implies $\partial_q h_v = \partial_p h_u$, so that $h_v = \partial_p g$ and $h_u = \partial_q g$ for some $g \equiv g(u, v, q, p)$. The final equation yields in terms of g

$$\partial_u \partial_p g - \partial_v \partial_q g + (\partial_p^2 g)(\partial_q^2 g) - (\partial_q \partial_q g)^2 = 0$$

which is Plebanski's second heavenly equation.□

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