Some Quaternionically Equivalent Einstein Metrics

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If M is a 4-manifold, not necessarily compact, admitting two Einstein metrics g_1, g_2 in the same conformal class such that the scalar curvature κ_1 of g_1 is non-zero while $\kappa_2 = 0$, then Brinkman [1] showed that M is conformally flat. This result may be restated quaternionically. By a quaternionic structure on a $4n$-manifold we mean a reduction of the structure group to $GL(n, \mathbb{H}) \times_{\mathbb{Z}/2} GL(1, \mathbb{H})$. This is equivalent to requiring that M has a rank three subbundle $\mathcal{G} \subset \operatorname{End} TM$ which locally has a basis I, J, K satisfying

$$I^2 = J^2 = -1 \quad \text{and} \quad IJ = K = -JI. \tag{$*$}$$

Now $GL(1, \mathbb{H})$ is isomorphic to $SU(2) \times \mathbb{R}_{>0}$, so when $n = 1$ we obtain the conformal group $CO(4) \cong GL(1, \mathbb{H}) \times_{\mathbb{Z}/2} GL(1, \mathbb{H})$ and Brinkman's result tells us about Einstein metrics with the same quaternionic structure.

If M^{4n}, $n \geqslant 2$, has a quaternionic structure and a compatible metric g, we may embed $\mathcal{G}$ in $\Lambda^2 T^*M$ by $I \mapsto \omega_I$, where $\omega_I(X, Y) = g(X, IY)$, and define a global 4-form Ω via the local formula $\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$. If Ω is parallel with respect to the Levi-Civita connection, M is said to be quaternionic Kähler. The fundamental example of such a manifold is quaternionic projective space $\mathbb{HP}(n)$ with its usual symmetric metric. Alekseevskiĭ [2] showed that quaternionic Kähler metrics are automatically Einstein and that the curvature tensor may be decomposed as

$$R = \lambda R_{\mathbb{HP}(n)} + R_0,$$

where λ is a constant positive multiple of the scalar curvature and $R_{\mathbb{HP}(n)}$ and R_0 have the symmetries of the curvature tensors of $\mathbb{HP}(n)$ and a hyperKähler metric, respectively. (For a hyperKähler manifold, $\mathcal{G}$ is trivialized by parallel complex structures satisfying ($*$).) If E and H are bundles associated to the basic representations of $Sp(n)$ and $Sp(1) \cong SU(2)$ on $\mathbb{C}^{2n}$ and $\mathbb{C}^2$, respectively, then $T_{\mathbb{C}}M \cong E \otimes_{\mathbb{C}} H$, $\mathcal{G}$ is the second symmetric power S^2H and $R_0 \in S^4E$.

Now suppose M^{4n} admits two metrics g^h, g^q with the same quaternionic structure such that g^h is hyperKähler (and hence scalar flat) and g^q is quaternionic Kähler with non-zero scalar curvature. Let ∇^h and ∇^q be the Levi-Civita connections of these metrics. The hyperKähler structure trivialises H and we obtain a section $h \in \Gamma(H)$ with $\nabla^h h = 0$. Since the twistor operator $D\colon H \xrightarrow{\nabla} H \otimes T^* \cong E \oplus (E \otimes S^2H) \to E \otimes S^2H$ is quaternionically invariant [3], $\nabla^q h = e$ is a section of E. Let $\tilde{h} = jh$, $\tilde{e} = je$ and consider the vector field $X = e\tilde{h} - \tilde{e}h$. Now $T \otimes T^*$ decomposes as

$$S^2H + S^2E + \mathbb{R} + \Lambda_0^2 E + S^2H\Lambda_0^2E + S^2HS^2E$$

and vector fields whose covariant derivatives lie in the first two terms are Killing, whilst those with derivatives in the first four summands give rise to quaternionic transformations. A simple calculation shows that X is quaternionic, while $i(e\tilde{h} + \tilde{e}h)$, $eh + \tilde{e}\tilde{h}$ and $i(eh - \tilde{e}\tilde{h})$ are Killing vectors. In the hyperKähler structure these are just IX, JX and KX and together with X they define a local action of the group $\mathbb{H}^*$ in which the compact subgroup $Sp(1) \cong SU(2)$ acts isometrically, but permutes I, J and K. A Weitzenböck argument now shows that R_0 lies in $S^4E^\perp$, where $E^\perp$ is the orthogonal complement in E to the span of e and $\tilde{e}$; so the orbits of the $\mathbb{H}^*$-action are flat in the hyperKähler structure. Since the vector fields span a quaternionic subspace, an argument of Gray [4] also shows that the 4-dimensional orbits are totally geodesic (with respect to either metric).

Given a hyperKähler $(4n+4)$-manifold N admitting such an $\mathbb{H}^*$-action which is free, we can construct a quaternionic Kähler manifold M^{4n} as follows. Fix a complex structure I and let $U(1)$ be the subgroup of $Sp(1)$ preserving I (and hence permuting J and K). Let $\mu\colon N \to \mathfrak{u}(1)^* \cong \mathbb{R}$ be a Kähler moment map for this action. The level sets of μ are actually $Sp(1)$-invariant and $M = \mu^{-1}(x)/\,Sp(1)$ is a quaternionic Kähler manifold [5]. Letting $\mathbb{H}^*$ act diagonally on $N \times \mathbb{H}$ gives a quaternionic Kähler metric on N in the same quaternionic class as the original hyperKähler metric. One may construct examples of such manifolds N as bundles over quaternionic Kähler manifolds, obtaining explicit formulae for both metrics. These constructions generalise the fibration $\mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n)$. In this flat case the quaternionic Kähler metric obtained on the total space is induced by the inclusion $\mathsf{H}^{n+1} \hookrightarrow \mathsf{HP}(n+1)$.

Kronheimer [6] shows that every adjoint orbit of nilpotent elements in a complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ has a hyperKähler metric. One may check that this structure admits an H^*-action of the type described above. If $\mathfrak{g}$ is simple, the smallest orbit fibres over a compact homogeneous quaternionic Kähler manifold and the classification in [2] shows that all such quaternionic Kähler metrics arise this way. When $\mathfrak{g} = \mathfrak{su}(3)$, the quaternionic Kähler manifold is $\mathsf{CP}(2)$ and locally one has a non-flat hyperKähler structure on the negative spin bundle (away from the zero section).

The moment map μ is actually a hyperKähler potential, that is μ is simultaneously a Kähler potential for each of the complex structures on N. A hyperKähler manifold N admits such a function if and only if it admits an H^*-action of the type described above. Also, μ is a hyperKähler potential for N if and only if the hyperKähler metric is given by

$$\nabla^2 \mu = g^h.$$

Further details will appear elsewhere.

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