A Non-Hausdorff Mini-twistor Space

This note is about another example of a non-Hausdorff complex manifold arising naturally in twistor theory. A mini-twistor space $\mathbb{T}$ is the 4-real-dimensional space of directed geodesics of a 3-real-dimensional Weyl space $\mathbb{W}$, which becomes a 2-complex-dimensional manifold if the Weyl space satisfies the Einstein-Weyl condition. Since it is defined as a space of geodesics, and geodesics can wind around in funny ways, a mini-twistor space is always liable to be non-Hausdorff. I will describe an example of a particularly simple Einstein-Weyl space where the mini-twistor space can be seen to be non-Hausdorff in a fairly tame way.

Recall first that a Weyl space $\mathbb{W}$ is a manifold with a symmetric connection D and a conformal metric [g] which is preserved by D. Given a choice g_{ab} of representative metric, the compatibility between conformal metric and conection means that we can define D in terms of the metric connection and a 1-form ω_a. Under change-of-choice of representative metric we have

$$g_{ab} \rightarrow \Omega^2 g_{ab} \;;\quad \omega_a \rightarrow \omega_a + 2\nabla_a \log \Omega \qquad (1)$$

so that we can think of a Weyl space as the pair (g_{ab}, ω_a) subject to (1). (For more details see e.g. [H],[JT],[PT]).

The connection D has a Riemann tensor and a Ricci tensor, but the Ricci tensor is not necessarily symmetric. The Einstein-Weyl condition on $\mathbb{W}$ is that the symmetrised Ricci tensor be proportional to the (conformal) metric. This can be written out as an equation on the Ricci tensor of the representative metric and the 1-form ω_a. In 3 dimensions the equation is

$$R_{ab} - \tfrac{1}{2}\nabla_{(a}\omega_{b)} - \tfrac{1}{4}\omega_a\omega_b = \Lambda g_{ab}\,, \text{ some } \Lambda. \qquad (2)$$

This equation is, from its definition, conformally invariant and can be regarded as a conformally-invariant generalisation of the Einstein equations. Note that spaces conformal to Einstein spaces satisfy (2) since we can use (1) to eliminate ω_a. These examples can be recognised by the fact that ω_a is exact.

The example I want to consider comes about by conformally rescaling and making identifications on flat space. Take the metric and 1-form as

$$g = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,;\quad \omega = 0 \qquad (3)$$

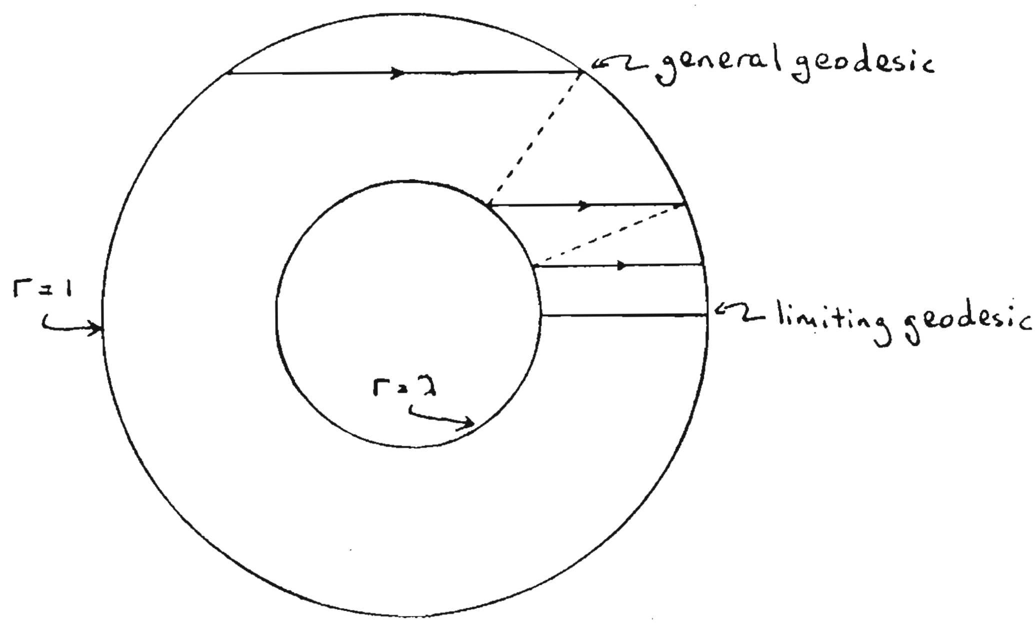
and conformally rescale with $\Omega = \exp(-\chi)$, defining $\chi = \log r$:

$$g = d\chi^2 + d\theta^2 + \sin^2\theta\, d\phi^2; \quad \omega = -2\,d\chi. \qquad (4)$$

Now impose a periodicity in χ to obtain an Einstein-Weyl structure on $S^1 x S^2$ (this example is given in [PT]; part of the interest of it is that this manifold has no Einstein metric). The periodicity in χ corresponds to identifying the radial coordinate r with λr for some λ with $0<\lambda<1$.

As I said at the beginning, the space of directed geodesics of a 3-dimensional Einstein-Weyl space **M** is a 2-dimensional complex manifold **Z**, the mini-twistor space of **M**. For flat space, the mini-twistor space is the space of directed lines in $\mathbb{R}^3$ which can be thought of as pairs of 3-dimensional real vectors (**a**, **b**) where **a** is unit and **b** is orthogonal to **a**. Equivalently, this is TP1, the tangent bundle of the complex projective line. For the example to be considered here we shall need to modify this a little.

A geodesic in the $S^1 x S^2$ Einstein-Weyl structure is as shown below:



It is basically a straight-line which, when it hits the outer sphere at r=1 is brought back to the inner sphere at $r=\lambda$ making the same angle with the radius vector. This means that in the future, the geodesic tends to a limiting one which is radially outwards and closed, while in the past it tends to a limiting one which is radially inwards and closed. In particular, this means that there are 'shadows' in the space: given a point p, points on the diameter through p but on the other side cannot be reached by geodesics through p (I am grateful to Paul Gauduchon for the suggestion that there might be shadows in this example). We shall return to these shadows below.

To construct the mini-twistor space **X**, consider first the closed radial geodesics. These correspond to the zero-section of **TP1**, ie to lines in $\mathbb{R}^3$ defined by pairs of the form $(a,0)$, but there are 2 closed radial geodesics for each radial geodesic in flat-space so we need to double the zero-section. Next the non-radial geodesics: think of a line in flat-space as a pair (a,b) then the process of bringing this back from the outer sphere to the inner sphere in the figure above corresponds to leaving a alone but rescaling b, $b \to \lambda b$, with λ as before.

This is then the mini-twistor space: delete the zero-section from TP1; identify b with λb in the fibres; then put two copies of the zero-section back. It is non-Hausdorff at the radial geodesics ie at the doubled-up zero-section, since any geodesic which is 'near to' a radially ingoing one is also 'near to' the continuation of it to the other side as a radially outgoing one.

A point p in the Einstein-Weyl space is represented by a holomorphic curve (a 'twistor line') in the mini-twistor space. The specification of this twistor line includes, at some stage, a choice of which of a pair of doubled-up points to take. Then any twistor line through the other of the relevant pair of doubled-up points in the mini-twistor space will correspond to a point of the Einstein-Weyl space in the 'shadow' of p.

A more complicated example of a non-Hausdorff mini-twistor space should be provided by the 'Berger sphere' Einstein-Weyl space, [JT],[HT]. This corresponds to a left-invariant metric on the 3-sphere. There is a special set of geodesics like the radial ones in the example above with the property that any other geodesic tends to one of them in the future and another in the past. The mini-twistor space seems to be a sort of deformed quadric with non-Hausdorff-ness along two generators of the same family. Henrik Pedersen and I have a description of it as a 'weighted projective space' but it is a little obscure.

(Like my other TN article, the work for this was done during a most pleasant visit to Henrik Pedersen in Odense, and I gratefully acknowledge hospitality received.)

[H] N.Hitchin in Twistor geometry and non-linear systems proceedings. Primorsko Bulgaria, 1980 ed. Doebner and Palev, Springer Lecture Notes in Mathematics 970

[JT] P.E.Jones and K.P.Tod CQG 2 (1985) 565

[PT] H.Pedersen and K.P.Tod Three-dimensional Einstein-Weyl geometry, to appear in Adv.in Math.

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