More on harmonic morphisms

I wish to correct two errors in my last Twistor Newsletter article, to make some observations which might make that article clearer, and to describe what I think is a new way of looking at the Kerr theorem appropriate to Riemannian twistor theory (though in this last connection see Hughston and Mason CQG 5 (1988) 275).

The two errors are as follows: firstly, the generator of the congruence should have been

$$L = \frac{1 - \alpha \bar{\alpha}}{1 + \alpha \bar{\alpha}} \frac{\partial}{\partial z} + \frac{\alpha + \bar{\alpha}}{1 + \alpha \bar{\alpha}} \frac{\partial}{\partial x} - i \frac{\alpha - \bar{\alpha}}{1 + \alpha \bar{\alpha}} \frac{\partial}{\partial y}$$

(this expression had some bars missing before)

and secondly $\alpha(x,y,z)$ is actually given implicitly by

$$f(x(1-\alpha^z) + iy(1+\alpha^z) - 2az, \alpha) = 0$$

(this expression had some signs muddled before).

In the interest of clarity, I should also have added that:

- if $\alpha = u + iv$ then
  $$\nabla^2 u = \nabla^2 v = 0 = \nabla u \cdot \nabla v ; |\nabla u|^2 = |\nabla v|^2$$

ie the complex function $\alpha(x,y,z)$ obtained from (2) defines the harmonic morphism,

- $\alpha(x,y,z)$ is constant along $L$ as given by (1),

- for a given constant value of $\alpha$, the real and imaginary parts of (2) each define a plane and $L$ is then tangent to the line of intersection of the two planes.

There is an 'elementary' interpretation of (2) as follows: a line in $\mathbb{R}^3$ is given by an equation of the form

$$r^T a = b$$

(3)

where $a$ is a unit vector and $b$ is orthogonal to $a$.

Parametrise the unit vectors with the complex number $\alpha$ according to (1), then (3) can be written

$$x(1-\alpha^z) + iy(1+\alpha^z) - 2az = \beta$$

(4)

where $\beta$ parametrises the Argand plane orthogonal to $a$. Now (2) is equivalent to taking $\beta$ to be an arbitrary holomorphic function of $\alpha$ ie the 'intercept' is an arbitrary holomorphic function of the 'direction', both understood as complex variables. When this function is linear, the

'Harmonic morphisms' is also the answer to the question 'what do you get from the Kerr theorem in Riemannian twistor theory (when there are no shear-free geodesic congruences) ?' To see this, take a homogeneous holomorphic twistor function \( F(Z^e) \) and intersect the zero-set of \( F \) with a line in twistor space which is 'real' in the sense appropriate to Riemannian twistor theory. This gives

\[
F(a+b\zeta, -b+a\zeta, 1, \zeta) = 0
\]

writing \((1, \zeta)\) for the \( \pi \)-spinor (rather than \((1, a)\) which I used at the beginning). Here \( a \) and \( b \) are complex coordinates on \( \mathbb{R}^4 \) and the metric is

\[
ds^2 = dad\bar{a} + dbd\bar{b}
\]

Solving (5) gives a function \( \zeta(a, b, \bar{a}, \bar{b}) \) with

\[
\frac{\partial \zeta}{\partial a} + \frac{\partial \bar{\zeta}}{\partial \bar{b}} = 0; \quad \frac{\partial \bar{\zeta}}{\partial b} - \frac{\partial \zeta}{\partial a} = 0
\]

from which it follows that \( \zeta \) has vanishing Laplacian and null gradient in the metric (6):

\[
\frac{\partial^2 \zeta}{\partial a \partial \bar{a}} + \frac{\partial^2 \bar{\zeta}}{\partial b \partial \bar{b}} = 0; \quad \frac{\partial^2 \bar{\zeta}}{\partial b \partial a} + \frac{\partial^2 \zeta}{\partial a \partial b} = 0
\]

If we think of \( \zeta \) as the stereographic coordinate on the sphere, then (8) is easily seen to be the conditions for (5) to define a harmonic morphism from \( \mathbb{R}^4 \) to \( S^2 \). However, (7) is stronger in that it implies that, as well as defining a harmonic morphism, \( \zeta \) is constant on flat 2-planes. Note that if \( \zeta \) satisfies (8) then so does any holomorphic function of \( \zeta \). In this sense, \( \zeta \) defines a family of holomorphically-related harmonic morphisms constant on flat 2-planes, one of which satisfies (7).

For the converse suppose that \( \eta \) satisfies (8) and is constant on flat 2-planes. Define \( \zeta \) by

\[
\frac{\partial \eta}{\partial \bar{a}} + \frac{\partial \bar{\eta}}{\partial \bar{b}} = 0 \quad \text{so that also} \quad \frac{\partial \bar{\eta}}{\partial b} - \frac{\partial \eta}{\partial a} = 0
\]

then it follows from the conditions on \( \eta \) that \( \zeta \) is a holomorphic function of \( \eta \) and so in turn satisfies (7) and (8).

To summarise, a family of holomorphically-related harmonic morphisms from \( \mathbb{R}^4 \) to \( S^2 \) which are constant on 2-planes defines and is defined by a holomorphic function in twistor space. This can be called 'the Riemannian Kerr theorem'.

(This view of the Kerr theorem arose in discussions with Henrik Pedersen.)

KPT