More on harmonic morphisms

I wish to correct two errors in my last Twistor Newsletter article, to make some observations which might make that article clearer, and to describe what I think is a new way of looking at the Kerr theorem appropriate to Riemannian twistor theory (though in this last connection see Hughston and Mason CQG 5 (1988) 275).

The two errors are as follows: firstly, the generator of the congruence should have been

$$L = \frac{1-\alpha\bar{\alpha}}{1+\alpha\bar{\alpha}}\frac{\partial}{\partial z} + \frac{\alpha+\bar{\alpha}}{1+\alpha\bar{\alpha}}\frac{\partial}{\partial x} - \frac{i(\alpha-\bar{\alpha})}{1+\alpha\bar{\alpha}}\frac{\partial}{\partial y} \qquad (1)$$

(this expression had some bars missing before)
and secondly $\alpha(x,y,z)$ is actually given implicitly by

$$f(x(1-\alpha^2) + iy(1+\alpha^2) - 2\alpha z, \alpha) = 0 \qquad (2)$$

(this expression had some signs muddled before).

In the interest of clarity, I should also have added that:

- if $\alpha = u+iv$ then
$$\nabla^2 u = \nabla^2 v = 0 = \nabla u \cdot \nabla v \;;\; |\nabla u|^2 = |\nabla v|^2$$
ie the complex function $\alpha(x,y,z)$ obtained from (2) defines the harmonic morphism,
- $\alpha(x,y,z)$ is constant along L as given by (1),
- for a given constant value of α, the real and imaginary parts of (2) each define a plane and L is then tangent to the line of intersection of the two planes.

There is an 'elementary' interpretation of (2) as follows: a line in $\mathbf{R}^3$ is given by an equation of the form

$$\mathbf{r}\wedge\mathbf{a} = \mathbf{b} \qquad (3)$$

where $\mathbf{a}$ is a unit vector and $\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Parametrise the unit vectors with the complex number α according to (1), then (3) can be written

$$x(1-\alpha^2) + iy(1+\alpha^2) - 2\alpha z = \beta \qquad (4)$$

where β parametrises the Argand plane orthogonal to $\mathbf{a}$. Now (2) is equivalent to taking β to be an arbitrary holomorphic function of α ie the 'intercept' is an arbitrary holomorphic function of the 'direction', both understood as complex variables. When this function is linear, the congruence is a 3-dimensional picture of the Kerr congruence and is sketched in Baird and Wood *Math. Ann. 280 (1988) 579-603.*

'Harmonic morphisms' is also the answer to the question 'what do you get from the Kerr theorem in Riemannian twistor theory (when there are no shear-free geodesic congruences)?' To see this, take a homogeneous holomorphic twistor function $F(Z^\alpha)$ and intersect the zero-set of F with a line in twistor space which is 'real' in the sense appropriate to Riemannian twistor theory. This gives

$$F(a+b\zeta, -\bar{b}+\bar{a}\zeta, 1, \zeta) = 0 \qquad (5)$$

writing $(1,\zeta)$ for the π-spinor (rather than $(1,\alpha)$ which I used at the beginning). Here a and b are complex coordinates on $\mathbf{R}^4$ and the metric is

$$ds^2 = da\,d\bar{a} + db\,d\bar{b} \qquad (6).$$

Solving (5) gives a function $\zeta(a,b,\bar{a},\bar{b})$ with

$$\frac{\partial\zeta}{\partial\bar{a}} + \zeta\frac{\partial\zeta}{\partial\bar{b}} = 0; \quad \frac{\partial\zeta}{\partial b} - \zeta\frac{\partial\zeta}{\partial a} = 0 \qquad (7)$$

from which it follows that ζ has vanishing Laplacian and null gradient in the metric (6):

$$\frac{\partial^2\zeta}{\partial a\partial\bar{a}} + \frac{\partial^2\zeta}{\partial b\partial\bar{b}} = 0; \quad \frac{\partial\zeta\partial\zeta}{\partial a\partial\bar{a}} + \frac{\partial\zeta\partial\zeta}{\partial b\partial\bar{b}} = 0 \qquad (8)$$

If we think of ζ as the stereographic coordinate on the sphere, then (8) is easily seen to be the conditions for (5) to define a harmonic morphism from $\mathbf{R}^4$ to S^2. However, (7) is stronger in that it implies that, as well as defining a harmonic morphism, ζ is constant on flat 2-planes. Note that if ζ satisfies (8) then so does any holomorphic function of ζ. In this sense, ζ defines a family of holomorphically-related harmonic morphisms constant on flat 2-planes, one of which satisfies (7).

For the converse suppose that η satisfies (8) and is constant on flat 2-planes. Define ζ by

$$\frac{\partial\eta}{\partial\bar{a}} + \zeta\frac{\partial\eta}{\partial\bar{b}} = 0 \quad \text{so that also} \quad \frac{\partial\eta}{\partial b} - \zeta\frac{\partial\eta}{\partial a} = 0$$

then it follows from the conditions on η that ζ is a holomorphic function of η and so in turn satisfies (7) and (8).

To summarise, a family of holomorphically-related harmonic morphisms from $\mathbf{R}^4$ to S^2 which are constant on 2-planes defines and is defined by a holomorphic function in twistor space. This can be called 'the Riemannian Kerr theorem'.

(This view of the Kerr theorem arose in discussions with Henrik Pedersen.)

KPT