An algorithm for the Penrose Transform

There is an algorithm for computing the E_1-term of the Penrose transform for homogeneous bundles between suitable domains in complex homogeneous spaces (see [1] for what this means) which is surprisingly simple, related to the topology of the homogeneous spaces and to Hecke modules. The significance is still a little unclear, but the hope is that the algorithm will allow us to establish precisely what maps can occur as higher differentials in the remainder of the hypercohomology spectral sequence. We illustrate for the ordinary twistor space $\leftrightarrow$ Minkowski space correspondence.

FACT 1 $\mathbb{PT}$ & $\mathbb{M}$ have certain diagrams attached which (1) record the topology of the spaces (by Morse strata) (2) record the possible (regular) homogeneous bundles on these spaces.

$$\mathbb{PT}: \quad \mathcal{O}_{\mathbb{P}} \xrightarrow{1} \Omega^1_{\mathbb{P}} \xrightarrow{2} \Omega^2_{\mathbb{P}} \xrightarrow{3} \Omega^3_{\mathbb{P}}$$

$$\mathbb{M}: \quad \begin{tikzcd} & & \Omega^2_+ \arrow[dr, "1"] & & \\ \mathcal{O} \arrow[r, "2"] & \Omega^1 \arrow[ur, "3"] \arrow[dr, "1"'] & & \Omega^3 \arrow[r, "2"] & \Omega^4 \\ & & \Omega^2_- \arrow[ur, "3"'] & & \end{tikzcd}$$

} HASSE diagrams with trivial ∞-char bundles

FACT 2 The E_1 term for $\Omega^3_{\mathbb{P}}$ is $\begin{array}{|ccc} \Omega^2_+ & \Omega^3 & \Omega^4 \\ 0 & 0 & 0 \\ \hline \end{array}$

(the E_1 term for the extreme rhs space in each Penrose transform is easy to compute — it's always a row, as high as possible)

FACT 3 Define a polynomial $E_X = \Sigma E_1^{p,q} u^p v^q$ for $E_1^{p,q}$ the first term in the Penrose transform of X on $\mathbb{PT}$ (so $X = \Omega^2_{\mathbb{P}}$ eg.) This is just a way of keeping track of the terms in E_1. If Y is a bundle in the diagram for $\mathbb{PT}$ or $\mathbb{M}$ define

(i) $T_i Y = v^{-1} Y$ if edge i is not incident with Y

(ii) $T_i Y = Z$ if $Z \xrightarrow{i} Y$

Then any bundle on $\mathbb{PT}$ is got by applying successive T_i to $\Omega^3_{\mathbb{P}}$ & we have, if $Z \xrightarrow{i} Y$: $\boxed{E_Z = T_i E_Y}$

Example $E_{\Omega^2_{\mathbb{P}}} = T_3 E_{\Omega^3_{\mathbb{P}}}$: $E^1_{\Omega^2_{\mathbb{P}}} = \begin{array}{|ccc} \Omega^1 & \Omega^2_- & 0 \\ 0 & 0 & \Omega^4 \\ \hline \end{array}$

$E^1_{\Omega^1_{\mathbb{P}}} = T_2 E^1_{\Omega^2_{\mathbb{P}}} = \begin{array}{|ccc} \mathcal{O} & 0 & 0 \\ 0 & \Omega^2_- & \Omega^3 \\ \hline \end{array}$ $\quad E^1_{\mathcal{O}_{\mathbb{P}}} = T_1 E^1_{\Omega^1_{\mathbb{P}}} = \begin{array}{|ccc} \mathcal{O} & \Omega^1 & \Omega^2_+ \\ \hline \end{array}$

Remarks i) T_i generate a Hecke algebra (deformation of group algebra of S_4); such play a key role in the famous Kazhdan–Lusztig conjecture.
ii) The recipe works for all groups/parabolics & gives an (Euler) character for all the cohomology groups which is easily computable. This is interesting in itself to representation theorists.

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[1] Baston/Eastwood "The Penrose Transform" OUP. 1989.