The Penrose Transform without spectral sequences.

On twistor space there are three short exact sequences connecting local twistor bundles and the differential forms. Thus

$$\begin{array}{c} 0 \to \Omega^3 \to \mathcal{T}^\alpha(-3) \to \Omega^2 \to 0 \\ 0 \to \Omega^2 \to \mathcal{T}^{[\alpha\beta]}(-2) \to \Omega^1 \to 0 \\ 0 \to \Omega^1 \to \mathcal{T}_\alpha(-1) \to \mathcal{O} \to 0. \end{array} \tag{1}$$

The maps are inherited from the *Koszul complex* for twistor space:

$$0 \to T^\alpha \to T^{[\alpha\beta]} \to T_\alpha \to \mathbf{C} \to 0,$$

and are given by $\omega \to Z^\alpha\omega$, $P^\alpha \to P^{[\alpha}Z^{\beta]}$, $Q^{[\alpha\beta]} \to \epsilon_{\alpha\beta\gamma\delta}Q^{\beta\gamma}Z^\delta$ and $R_\alpha \to R_\alpha Z^\alpha$. Similar sequences are available on any homogeneous space, where an analogue of a local twistor bundle is an extension of two irreducible bundles in a BGG resolution [1] linked by a simple reflection.

The easiest Penrose transform on any complex homogeneous space is always that of the highest forms—the result is non zero in highest possible degree only, always the kernel of an invariant differential operator, resolved by further invariant operators and irreducible (over $\mathrm{sl}(4, \mathbf{C})$ in the standard twistor case). Here of course we get self-dual Maxwell fields:

$$0 \to H^1(\Omega^3) \to \mathcal{O}_{A'B'}[-1] \xrightarrow{\nabla^{B'}_{A'}} \mathcal{O}_{AA'}[-3] \xrightarrow{\nabla^{AA'}} \mathcal{O}[-4] \to 0.$$

The naïve idea is to start from this and use the long exact sequences on cohomology coming from (1) to compute the cohomology of the rest. The first step is to compute the cohomology of the local twistor bundles and the second to use this in the long exact sequences.

Take the case of $\mathcal{T}^\alpha[-3]$. This is obtained by coupling the result for $\mathcal{O}[-3]$ to local twistor transport on Minkowski space, whilst the result for $\mathcal{O}[-3]$ is in turn obtained from $H^1(\Omega^3)$ by helicity lowering. Thus $H^1(\mathcal{T}^\alpha[-3])$ consists of solutions of the local twistor equation

$$0 = \nabla_A^{A'} \begin{pmatrix} \omega^B_{A'} \\ \pi_{A'B'} + \frac{1}{2}\epsilon A'B'\phi \end{pmatrix} = \begin{pmatrix} \nabla_A^{A'}\omega^B_{A'} - \delta^B_a\phi \\ \nabla_A^{A'}\pi_{A'B'} + \frac{1}{2}\nabla_{AB'}\phi \end{pmatrix}.$$

Here, $\omega_{AA'} \in \mathcal{O}_{AA'}$ is a one form, $\phi \in \mathcal{O}[-2]$ and $\pi_{A'B'} \in \mathcal{O}_{(A'B')}[-1]$ is a self dual two-form. The first of these equations simply fixes $\phi = \frac{1}{2}\nabla_A^{A'}\omega^A_{A'}$ and one is left with

$$\nabla^{A'}_{(A}\omega_{B)A'} = d^+\omega = 0 \quad \nabla_A^{A'}\pi_{A'B'} - \frac{1}{4}\nabla_{AB'}\nabla^c\omega_c = 0.$$

Since $d : \Omega^2_+ \to \Omega^3$ is onto $\ker d : \Omega^3 \to \Omega^4$, we can regard this last equation as fixing $\pi_{A'B'}$ (up to a self dual Maxwell field) and requiring $\Box\nabla^c\omega_c = 0$. Thus

$$H^1(\mathcal{T}^\alpha(-3)) = \text{ s.d. Maxwell } + \{\omega_c | d^+\omega = 0 = \Box\nabla^c\omega_c\}.$$

$H^1(\Omega^2)$ is then just the second term. This is a non trivial extension of two irreducibles, namely the anti-self-dual Maxwell fields (obtained via potentials mod gauge) and

$$\{f \in \mathcal{O} \mid \Box^2 f = 0)/\mathbf{C}$$

(obtained by letting $\omega_c = \nabla_c f$). Of course, this calculation checks with more standard methods. The cohomology of $\Omega^1, \mathcal{O}$ are equally easy to compute this way.

To carry this out for general complex homogeneous spaces we have to overcome two difficulties. The first is to obtain the cohomology of local twistor bundles. This is done by observing that helicity raising and lowering (or the *translation principle*) commutes with taking cohomology and that the result splits into a direct sum of (χ-primary) parts under the action of the center of $\mathcal{U}(\mathbf{g})$—i.e. under Casimir operators. One has to perform two translations, by a finite dimensional representation and its dual. The first translates to *singular* character and the second out again to regular character. Each time, we also project out only one of the χ-primary parts. Let's label these two operations ψ_α, ϕ_α—α is a simple root for $\mathbf{g}$ and the finite dimensional module is just $F(\lambda)$ where λ is the fundamental weight dual to α. Vogan has an algorithm for calculating the $\phi_\alpha\psi_\alpha$ on irreducibles, which succeeds by the Kazhdan–Lusztig conjectures. This reduces the problem to (not too difficult) combinatorics. The second difficulty is to understand the maps in the long sequence. Here, translation again comes to the rescue. It turns out that we can often pick α so that for a given homogeneous bundle $\mathcal{F}$, $\psi_\alpha\mathcal{F} = 0$. Then $\psi_\alpha H^\bullet(\mathcal{F}) = 0$ too. So any irreducible *not* annihilated by ψ_α can't occur $H^\bullet(\mathcal{F})$! This data is easy to deduce from Hasse diagrams (see my other article in this TN).

Conjecture: This is enough (with Schur's lema) to calculate the maps in the long exact sequence (and so to compute the Penrose transform of any homogeneous bundle).

I've checked this to be true in many cases, one involving so(12)!

A slightly stronger conjecture is that if an irreducible occurs in H^i and $\phi_\alpha\psi_\alpha H^i$ then it is mapped non trivially from one to the other. This is true in all the examples I know. If it is true in general it should follow that the Penrose transform will detect all non-trivial homomorphisms of Verma modules.

[1] Baston/Eastwood: *The Penrose transform—its interaction with representation theory*. O.U.P. (1989).

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