Cohomological contours and cobord maps

Introduction

In [2], we asked how to tell whether a given contour κ is cohomological, that is, whether κ treats the twistor functions on the 'outside' part of the diagram properly as cohomology classes. If A and B are a pair of ears in a twistor diagram (i.e. a pair of planes on which a twistor function blows up) then by duality, κ is cohomological with respect to (A, B) iff $\kappa = \partial_* \lambda$ where ∂_* is the Mayer-Vietoris connecting homomorphism from the complement of $A \cup B$ to the complement of $A \cap B$. By the main Lemma of [2] (reproduced as Lemma 1 below), this somewhat impractical criterion is equivalent to the following

> CRITERION: κ is cohomological if *either* there is a λ with
>
> $$\delta_a(\lambda) = \kappa \text{ and } \delta_b(\lambda) = 0$$
>
> *or* there is a μ satisfying these conditions with a and b interchanged.

Here $\delta_{a,b}$ are the cobord maps corresponding to A, B. The reason this is useful is that the cobord maps are dual to the external S^1 integrals which are used all the time in twistor diagram theory.

APH has suggested that for any twistor diagram whose evaluation does not require the use of a contour with boundary, one has the much stronger $\kappa = \delta_a \delta_b(\text{something})$. Moreover, he has an example of a cohomological contour (for the scalar product diagram for spin 1) whose evaluation requires the use of a boundary contour (or some other such device) for which the *relative* version of the above criterion holds, but for which the condition of this paragraph *fails*.

In this article we show that the criterion of [2] does indeed carry over to the relative situation and we also prove APH's conjecture about contours without boundary by relating it to the main result of [1]. We make significant use of the lemma mentioned above, which we restate here in a generalized form:

Lemma 1 *Let X be a complex manifold and let S_1, S_2 be complex submanifolds of (complex) codimensions p_1, p_2, in general position so that $S = S_1 \cap S_2$ is a complex submanifold of complex codimension $p = p_1 + p_2$.*

Consider the two Leray sequences

$$\begin{array}{ccccccccc}
\to & H_{i+1}(X - S) & \xrightarrow{\cap_a} & H_{i-2p_1+1}(S_1 - S_2) & \xrightarrow{\delta_a} & H_i(X - S_1) & \to \\
 & \downarrow & & \| & & \downarrow & \\
\to & H_{i+1}(X - S_2) & \xrightarrow{\cap_b} & H_{i-2p_1+1}(S_1 - S_2) & \xrightarrow{\delta_b} & H_i(X - S_1 - S_2) & \to
\end{array}$$

Then the composite $\delta_b \cap_a$ is equal to the Mayer-Vietoris connecting homomorphism

$$\partial_* : H_{i+1}(X - S) \longrightarrow H_i(X - S_1 - S_2).$$

Proof. This is exactly as in [2]: the essential point is that even though we have allowed the codimensions to exceed 1, the formula for the cobord map in terms of compactly supported differential forms is unchanged. □

Residues, cobord maps and the 'general procedure'

In [1] it was shown that if κ is a cohomological contour without boundary which evaluates a projective twistor diagram then κ is necessarily in the image of the cobord map

$$\Delta : H_v(\Lambda - \Sigma) \longrightarrow H_{6v-1}(\Pi - \Lambda - \Sigma) \qquad (1)$$

where v is the number of fields, Π is the product of v (projective) twistor/dual twistor spaces, Λ is the product of lines on which the fields are based and Σ is the singularity set of the kernel (interior) of the diagram. An element κ in the RHS is used to evaluate the diagram by *dotting* all the external fields together, multiplying by the kernel of the diagram and integrating over κ.

On the other hand, those who evaluate diagrams on a professional basis are accustomed to the use of a large number of 'small' (i.e. lower codimensional) cobord maps. We remarked in [1] that the methods had to be essentially equivalent and dual to the relationship between cupped and dotted forms. Using Lemma 1, we can now make this precise.

Proposition 1 *Let X, S_1, S_2 etc. be as in Lemma 1. Then the following diagram is commutative:*

$$\begin{array}{ccccc} H_*(S) & \xrightarrow{\delta_2} & H_{*+2p_2-1}(S_1 - S_2) & \xrightarrow{\delta_1} & H_{*+2p-2}(X - S_1 - S_2) \\ \| & & & & \| \\ H_*(S) & \xrightarrow{\Delta} & H_*(X - S) & \xrightarrow{\partial_*} & H_{*+2p-2}(X - S_1 - S_2) \end{array}$$

Here $\Delta, \delta_1, \delta_2$ are the obvious cobord maps and ∂_* is the Mayer-Vietoris connecting homomorphism.

Proof. By Lemma 1, ∂_* can be replaced by the composite

$$H_{*+2p-1}(X - S) \xrightarrow{\cap_a} H_{*+2p-1-2p_1}(S_1 - S_2) \xrightarrow{\delta_b} H_{*+2p-2}(X - S_1 - S_2).$$

Since $\delta_b = \delta_1$ (and $p - p_1 = p_2$) it is enough to show that the following diagram is commutative:

$$\begin{array}{ccc} H_*(S) & \xrightarrow{\delta_2} & H_{*+2p_2-1}(S_1 - S_2) \\ \downarrow \Delta & & \| \\ H_{*+2p-1}(X - S) & \xrightarrow{\cap_a} & H_{*+2p_2-1}(S_1 - S_2) \end{array}$$

But that is obvious. □

Theorem 1 *Suppose that the above twistor diagram with v (external) vertices has an evaluation by means of a contour without boundary. Then any evaluation coming from the 'general procedure' (1) can be implemented by integration over $2v$ circles (one for each ear) followed by integration over a contour in $\Lambda - \Sigma$.*

Proof. (i) By applying the Proposition to the case $X = CP^3$, each S_i equal to a plane, we conclude that the cobord associated to a line in CP^3 is the same as the composition of two cobords, one for each plane, and a Mayer-Vietoris map.

(ii) Similarly, by applying the Proposition with X equal to the product of v twistor spaces and each S_i equal to the product of $v-1$ twistor spaces and one projective line, we see that the 'fat' cobord map Δ of (1) coincides with a cobord for each line and $v-1$ Mayer-Vietoris maps.

Combining (i) and (ii), we complete the proof.□

This theorem brutally exhibits the strengths and weaknesses of the 'general procedure'. It shows that *any cohomological evaluation by integration over a closed contour factors through taking residues at the external lines.* That is the strength of the procedure: its weakness is that this residue is very often zero for interesting twistor diagrams; for such diagrams, boundary contours or some non-trivial extension of the class of functionals considered in [1] are plainly needed.

A criterion for cohomological boundary contours

When is a boundary contour cohomological? As in the non-boundary case, a contour is cohomological if it lies in the image of a certain Mayer-Vietoris map. As we remarked above, this is not much use. However, with the aid of the following relative version of our basic lemma, we can get a practical criterion for a boundary contour to be cohomological.

Lemma 2 *Let X, S_1, S_2 etc. be as in Lemma 1. In addition, let F be a closed submanifold in general position. Consider the two Leray sequences*

$$\begin{array}{ccccccccc} \rightarrow & H_{i+1}(X-S,F) & \xrightarrow{\cap_a} & H_{i-2p_1+1}(S_1-S_2,F) & \xrightarrow{\delta_a} & H_i(X-S_1,F) & \rightarrow \\ & \downarrow & & \| & & \downarrow & \\ \rightarrow & H_{i+1}(X-S_2,F) & \xrightarrow{\cap_b} & H_{i-2p_1+1}(S_1-S_2,F) & \xrightarrow{\delta_b} & H_i(X-S_1-S_2,F) & \rightarrow \end{array}$$

Then the composite $\delta_b \cap_a$ is equal to the Mayer-Vietoris connecting homomorphism

$$\partial_* : H_{i+1}(X-S,F) \longrightarrow H_i(X-S_1-S_2,F). \qquad (2)$$

Proof. Recall that an element of $H_i(X, F)$ is represented by a pair of compactly supported forms (α, α') such that

$$\alpha \in \Lambda_c^{\dim X - i}(X),\ \alpha' \in \Lambda_c^{\dim X - i - 1}(F),\ d\alpha = 0,\ \alpha|F = d\alpha'. \qquad (3)$$

(Here and throughout, '$|F$' means 'pull-back to F'.)

Let D be a tubular neighbourhood of $S_1 - S_2$, relatively compact in $X - S_2$; let j be the inclusion of $S_1 - S$ in $X - S$; let $\pi : D \to S_1$ be the projection which we may suppose carries points of $F \cap D$ to points of $F \cap (S_1 - S_2)$. Then there exists a homotopy operator H such that

$$\pi^* j^* u - u = (dH + Hd)u \text{ and } (Hu)|F = H(u|F) \qquad (4)$$

for all forms u on D.

Let β be a positive smooth bump function on $X - S$, equal to 0 near $S_1 - S$ and equal to 1 in a neighbourhood of $X - D$. Then we have the following formulae:

$$\begin{aligned} \partial_*(\alpha, \alpha') &= (\alpha \wedge d\beta, \alpha' \wedge (d\beta|F)); \\ \cap_a(\alpha, \alpha') &= (j^*(\alpha), j^*(\alpha')); \\ \delta_b(\gamma, \gamma') &= (\pi^*(\gamma) \wedge d\beta, \pi^*(\gamma') \wedge (d\beta|F)). \end{aligned}$$

So

$$\begin{aligned} \delta_b \cap_a (\alpha, \alpha') &= (\pi^* j^*(\alpha) \wedge d\beta, \pi^* j^*(\alpha') \wedge (d\beta|F) \\ &= (\alpha \wedge d\beta + dH\alpha \wedge d\beta, \alpha' \wedge (d\beta|F) + Hd\alpha' \wedge (d\beta|F) + dH\alpha' \wedge (d\beta|F)) \\ &= \partial_*(\alpha, \alpha') + (dH\alpha, (H\alpha|F) + dH\alpha') \wedge d\beta \end{aligned}$$

using both parts of (4) and (3). To complete the proof, it suffices to note that the 'error term' in the equation above represents zero in relative homology by definition. □

Thus the criterion of TN28, restated in the Introduction, holds also for boundary contours: thus some of APH's standard procedures (reduction to CP^2 integrals) are justified.

What is still missing is a relative version of the 'general procedure' of [1]. Any offers, anyone?

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References

[1] Huggett & Singer, Projective twistor diagrams and relative cohomology, Trans. A.M.S. (to appear, at last, in November 1990).

[2] Huggett & Singer, Cohomological residues, TN28.