Some new boundary-contour integrals

These notes fill in some gaps in the theory of contours for twistor diagrams. First, there's a spinor integral with two boundaries which doesn't seem to have been studied before, namely:

$$\frac{1}{2\pi i}\oint_{\xi.\beta=0,\ \eta.\delta=0} \frac{D\xi \wedge D\eta}{(\xi.\eta)(\eta.\gamma)(\xi.\alpha)} \qquad (1)$$

To construct a contour, first take the special case $\alpha = \gamma$, and coordinates

$$\xi^A = x\alpha^A + \beta^A, \qquad \eta_A = y\alpha_A + \beta_A$$

so the integral becomes

$$\frac{1}{2\pi i}(\alpha.\beta)^{-1}\oint_{x=0,\ y\alpha.\delta+\beta.\delta=0} dx\, dy\, (x-y)^{-1}$$

In this form it's immediate that there is a contour with the topology of an annulus yielding

$$\frac{\beta.\delta}{\alpha.\beta\ \alpha.\delta}$$

By a power series expansion, or otherwise, the general case can be evaluated as

$$(\alpha.\gamma)^{-1}\log\left(\frac{\alpha.\beta\ \gamma.\delta}{\alpha.\delta\ \gamma.\beta}\right) = \int_0^1 \frac{\beta.\delta\ du}{\alpha.\gamma\,\beta.\delta\, u - \alpha.\delta\,\beta.\gamma}$$

The contour can be thought of as a sphere with two holes in it. One hole has boundary on $\xi.\beta = 0$, and allows the singularity in $\eta.\gamma$ to poke through it; similarly for the other. If one of these singularities is absent, the corresponding hole can be filled in with a 'cap' (which makes no difference to the answer.) Thus neither pole is essential to the contour. This can be seen explicitly by evaluating a different form over the same contour, viz.

$$\frac{1}{2\pi i}\oint_{\substack{\xi.\beta=0\\ \eta.\delta=0}} \frac{D\xi\wedge D\eta}{(\xi.\eta)^2}\left(\frac{\eta.\nu}{\eta.\gamma}\right)\left(\frac{\xi.\kappa}{\xi.\alpha}\right) = \frac{\nu.\beta}{\gamma.\beta}\frac{\delta.\kappa}{\delta.\alpha} + \frac{\alpha.\kappa\ \nu.\gamma}{\alpha.\gamma}\left(\frac{\beta.\delta}{\delta.\alpha\,\beta.\gamma} + \frac{1}{\alpha.\gamma}\log\left(\frac{\alpha.\beta\ \gamma.\delta}{\alpha.\delta\ \gamma.\beta}\right)\right)$$

Then in the cases where $\nu = \gamma$ or $\kappa = \alpha$ we can 'cap' one hole and recover spinor integrals already well-known*

$$\frac{1}{2\pi i}\oint_{\eta.\delta=0}\frac{D\xi\wedge D\eta}{(\xi.\eta)^2}\frac{\xi.\kappa}{\xi.\alpha} = \frac{\delta.\kappa}{\delta.\alpha}\ ; \qquad \frac{1}{2\pi i}\oint_{\xi.\beta=0}\frac{D\xi\wedge D\eta}{(\xi.\eta)^2}\frac{\eta.\nu}{\eta.\gamma} = \frac{\nu.\beta}{\gamma.\beta}$$

If both $\nu = \gamma$ and $\kappa = \alpha$
then both caps can be put on and we recover $\displaystyle \frac{1}{2\pi i}\oint \frac{D\xi\wedge D\eta}{(\xi.\eta)^2} = 1$

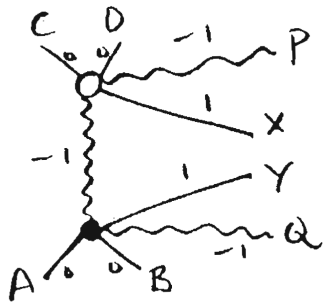
There is also an analogous contour for

$$\oint_{\substack{\zeta\cdot\beta=0,\ \eta\cdot\delta=0 \\ \zeta\cdot\eta=0}} \frac{D\zeta\wedge D\eta}{(\eta\cdot\gamma)^2(\zeta\cdot\alpha)^2} \qquad (2)$$

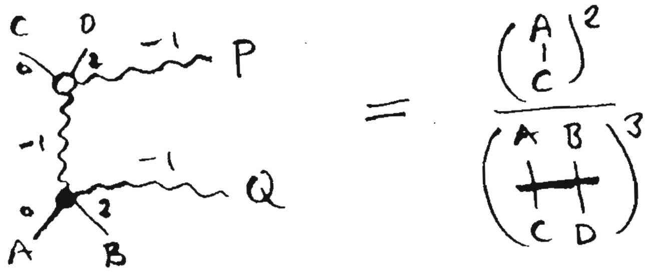
To see this, use the same special case and the same coordinates, then the contour is readily specified as the interior of a triangle bounded by the three given surfaces. The result in the general case is

$$\int_0^1 \frac{(\beta\cdot\delta)^2\, u\, du}{(\alpha\cdot\gamma\,\beta\cdot\delta\, u - \alpha\cdot\delta\,\beta\cdot\gamma)^2}$$

These contours for spinor integrals imply (via cobordism) contours for certain twistor integrals. In particular, (2) induces a contour for:



with the property that (AB), (CD) are treated cohomologically, whilst the X and Y poles are not essential.
Because these poles can be 'filled in', we can operate with $(X\cdot\partial_D)^2(Y\cdot\partial_B)^2$ and still have a non-zero integral: in fact explicit calculation yields



But the RHS is just the inner product for the two spin-1 elementary states. This means that the spin-1 inner product *can* be represented as a twistor integral which reduces to a spinor integral, i.e. with a contour that fibres as

$(S^1)^4 \times$ 

but to do this we have to introduce these extra boundaries.

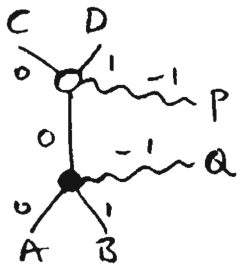
Since the position of the boundaries makes no difference to the answer, it looks as though they can be removed by adding some 'caps'; and indeed they can within the *entire* twistor integral, but at the cost of abandoning the fibration which reduces it to a spinor integral. In fact the contour (without any extra boundaries) can be specified as (solid ball in R^4 x $(S^1)^2$). This integral is of course the simplest example of a cohomological contour which cannot be realised as (a spinor integral x $(S^1)^4$) (see the article by SAH and MAS in this **TN**).

These new spinor integrals are also useful by guiding the analogous construction in CP^2. That is, we study

$$\oint_{\substack{b\cdot x=0\\ d\cdot y=0}} \frac{D^2x\wedge D^2y}{(x\cdot y)(c\cdot y)^2(a\cdot x)^2}\ ;\quad \oint_{\substack{b\cdot x=0\\ d\cdot y=0\\ x\cdot y=0}} \frac{D^2x\wedge D^2y}{(c\cdot y)^3(a\cdot x)^3}$$

where x^a, y_a are elements of CP^2, CP^{2*}.

The construction of contours is not quite as immediate as in the spinor case but can be done. We can then use them to induce contours for the twistor diagrams



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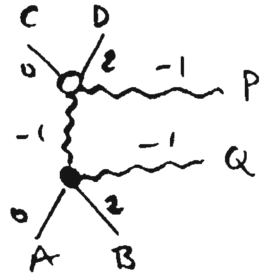

which are *not* cohomological in (AB), (CD). The first of these contours was shown to exist by SAH (D. Phil. thesis, 1980) but until now we haven't had a direct construction for it. The second is closely related. Explicit evaluation of the integrals shows agreement with the results previously derived by limiting techniques*). These contours are important in the theory of inhomogeneous diagrams*. There are further generalisations and deductions from these new constructions which, put together with SAH and MAS's recent work, should cap most of the holes in the theory of elementary twistor diagrams.

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*APH, Proc. R. Soc. Lond. A 397, 341-374 (1985)