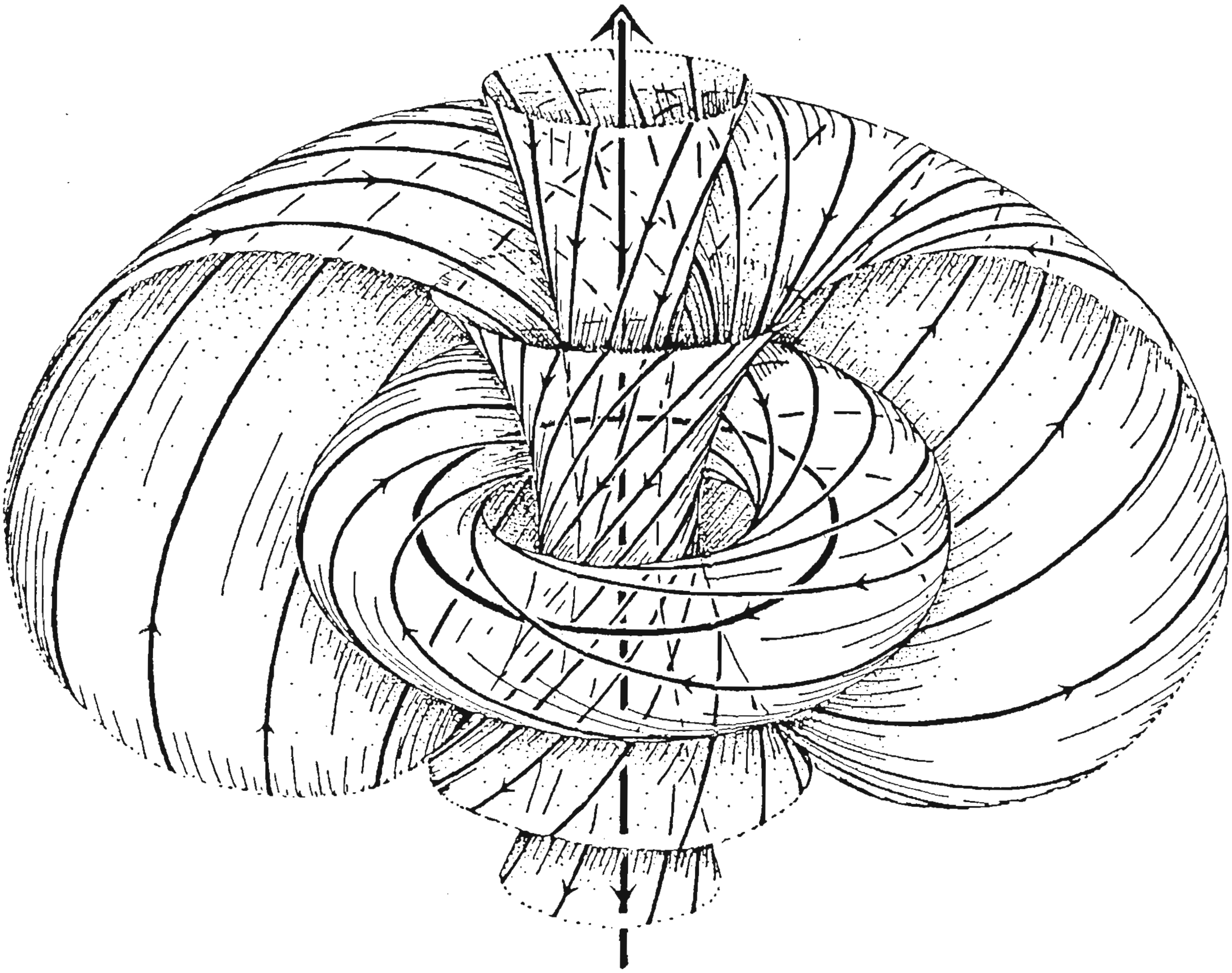
Twistor Newsletter (no 30: 18, June 1990)



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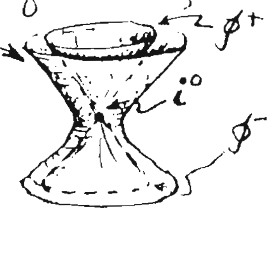
Light Rays Near i^0 : A New Mass-Positivity Theorem

As was emphasized by Helmut Friedrich in a recent survey talk [given at the E.T. Newman birthday conference, Pittsburgh, May 1990], one of the major problems obstructing a proper understanding of asymptotically flat space-times is the lack of a completely satisfactory geometrical framework for analysing i^0 (the conformally singular point at spacelike infinity). Perhaps twistor theory can make a significant contribution here. Whereas partial understandings have come through the work of Ashtekar-Hansen, Geroch, Beig, Sachs, Sommers and many others, there remains a distinct conceptual awkwardness, and as yet there is no elegant geometrical description to match — and to unite with — that of $\mathscr{I}^{\pm}$. The "spi" construction of Ashtekar-Hansen seems to come closest [J. Math. Phys. **19**(1978)1542] and there are indeed some relations to twistor theory. In Minkowski space $\mathbb{M}$, the points of spi can be identified with timelike hyperplanes (i.e. timelike 3-quadrics through i) and these are represented by (proportionality classes of) skew twistors $S^{\alpha\beta}$ subject to $S^{\alpha\beta}I_{\alpha\beta}=0$ and the "reality condition" $\bar{S}_{\alpha\beta}I^{\gamma\beta}=I_{\alpha\beta}S^{\gamma\beta}$. Such objects find their place in the <u>moment</u> <u>sequence</u> (see Spinors & Space-Time, vol. 2, pp 94, 96). The points of $\mathscr{I}$ arise when $S^{\alpha\beta}$ is <u>simple</u> (and so has the form $Z^{[\alpha}I^{\beta]\gamma}W_\gamma$).

I shall postpone, until a later date, a detailed discussion of the geometry involved here. In any case, we cannot expect to capture the structure of i^0 that incorporates the total mass-momentum and angular momentum of the system using merely Minkowski-space twistors. Possible schemes for involving more general types of twistor would be to use 2-surface twistors in the manner of Shaw, or perhaps to use hypersurface twistors, e.g. for a hyperboloidal timelike hypersurface that approximates $i^0 \cup \mathscr{I}^+ \cup \mathscr{I}^-$.



Of more immediate physical relevance would be to study the behaviour of light rays near i^0. There is a clear connection with twistor theory here, but for the moment I shall just show how this study can be used to give a new and perhaps comparatively simple proof of total energy positivity in general relativity (compare Schoen & Yau, Phys. Rev. Lett. **43** (1979) 1457, Commun. Math. Phys **65** (1979) 45; Witten, Commun. Math. Phys **80** (1981) 381):

Theorem If $\mathcal{M}$ is asymptotically simple and satisfies the strong null convergence condition, then it cannot have a well-defined negative mass at i^0.

N.B. By the "strong" null convergence condition (**SNCC**) I shall mean that every endless null geodesic contains a pair of conjugate points. This is a consequence of the three properties:

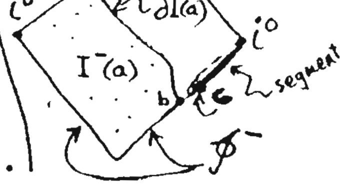
(1) $R_{ab}n^a n^b \leq 0$ for all null vectors n^a (i.e. with Einstein's equations, & my sign conventions, $T_{ab}n^a n^b \geq 0$),

(2) null geodesic completeness, and

(3) the genericity condition: $k_{[e}R_{a]bc[d}k_{f]}n^b n^c \neq 0$ somewhere along each null geodesic. [see Hawking & Penrose, Proc. Roy. Soc. (Lond.) A314 (1970) 529.]

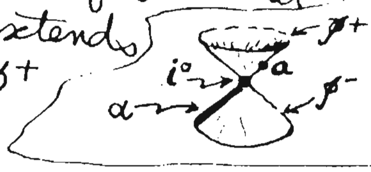
Outline of proof

The idea is to assume that the mass at i^0 is negative and that SNCC holds, and then to derive a contradiction. Let $a \in \mathscr{I}^+$ and consider $I^-(a)$. (All sets are in $\bar{\mathcal{M}} = \mathcal{M} \cup \mathscr{I}^+ \cup \mathscr{I}^-$; here "$I^-(a)$" is to include its limit points (interior limit points only, so that $I^-(a)$ remains open) on $\mathscr{I}^-$; "$\partial I^-(a)$" is supposed to be composed of the points of $\partial I^-(a)$ in $\mathcal{M}$, together with the limit points thereof in $\bar{\mathcal{M}}$.) Let us see what would happen if it could be shown that some segment of a generator of $\mathscr{I}^-$, from a point $c \in \mathscr{I}^-$ to i^0, has a neighbourhood in $\bar{\mathcal{M}}$ that does not meet $I^-(a)$ (— as follows if any $c \in \mathscr{I}^-$ has a neighbourhood not meeting $I^-(a)$).

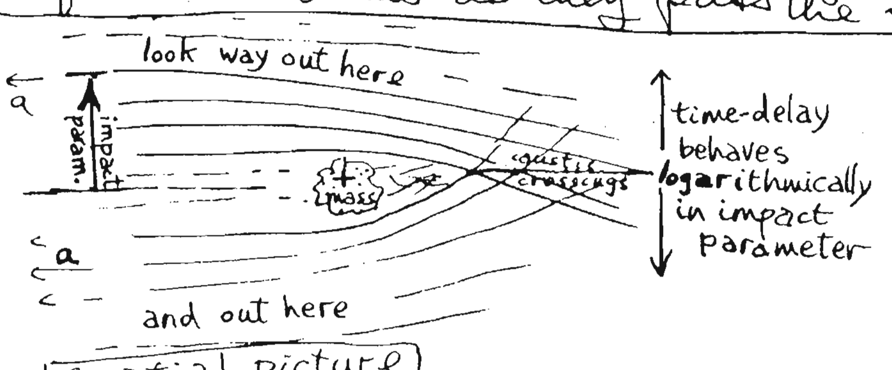


In that case $\partial I^-(a)$ would meet $\mathscr{I}^-$ transversely in some (non-vacuous) set $\mathscr{B}$ $(= \partial I^-(a) \cap \mathscr{I}^-)$. Let $b \in \mathscr{B}$. By standard theorems (since $\mathcal{M}$ is globally hyperbolic, as follows from its asymptotic simplicity) there must be a null geodesic from **b** to **a**, lying on $\partial I^-(a)$, with no pair of conjugate points between b and a, which contradicts SNCC.

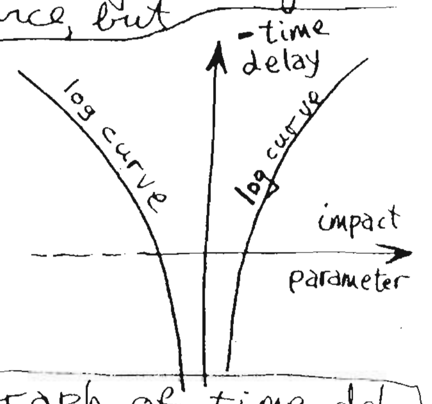
In fact there is <u>no</u> such segment (or point c), but what <u>does</u> happen, with negative mass at i^0, is almost as effective. Let us examine the generators of the past light cone $\mathcal{A}$ of **a** (we have $\partial I^-(a) \subset \mathcal{A}$), particularly in the neighbourhood of that generator α of $\mathscr{I}^-$ that is "diametrically opposite" to **a** (i.e. α extends through i^0 to become the generator of $\mathscr{I}^+$ containing **a**). We shall be mainly concerned with generators of $\mathcal{A}$ "close" to that particular generator of $\mathscr{I}^+$ on which **a** lies (i.e. α extended), and with their intersections with $\mathscr{I}^-$.



To assist us in picturing this situation, consider first the case of <u>positive</u> mass at i^0, to see why there is no conflict with SNCC. Not only are the light rays deflected inwards as they pass the source, but



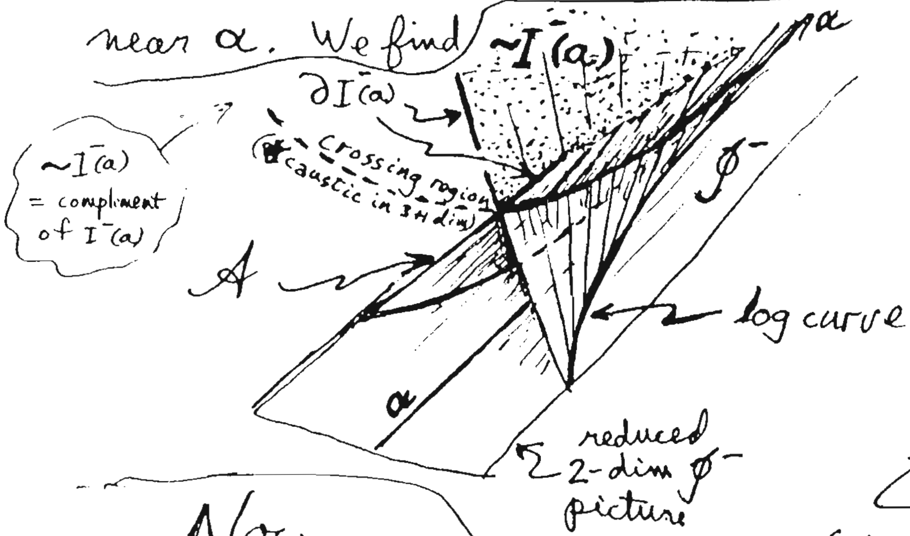
Spatial picture

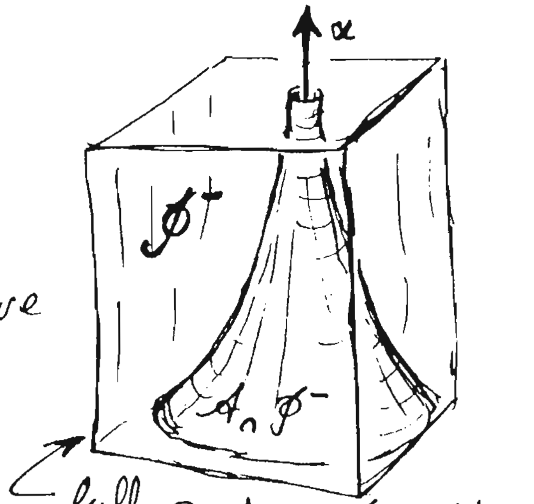


Graph of time-delay

there is also a time-delay that behaves logarithmically in the impact parameter (distance of "closest approach"). This has no natural zero, so the larger the impact param., the larger the values of advanced time that will eventually be encountered. This phenomenon was described in detail by R.P. in the Taub Festschrift volume [ed. F. Tipler, ~1980]. We find that the whole of $\mathscr{I}^-$ is contained in $I^-(a)$. (Hence $\partial I^-(a) \cap \mathscr{I}^- = \emptyset$, so no contradiction with SNCC.)

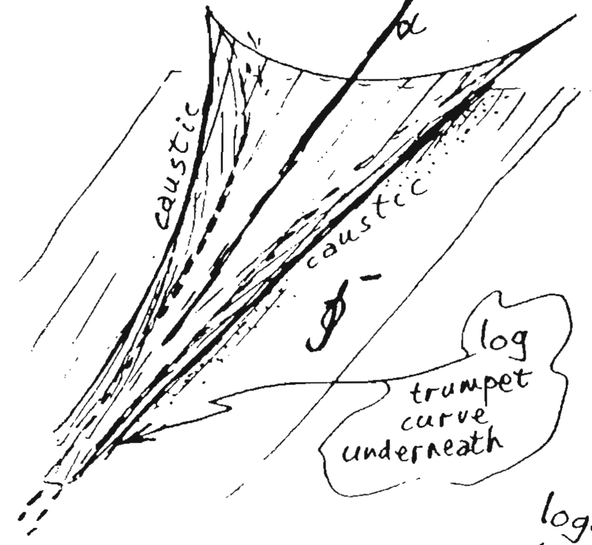
To see how this comes about, examine $\mathcal{A}$ and $\partial I^-(a)$ near α. We find

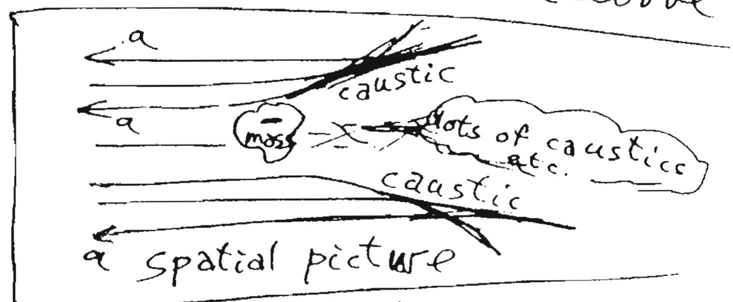


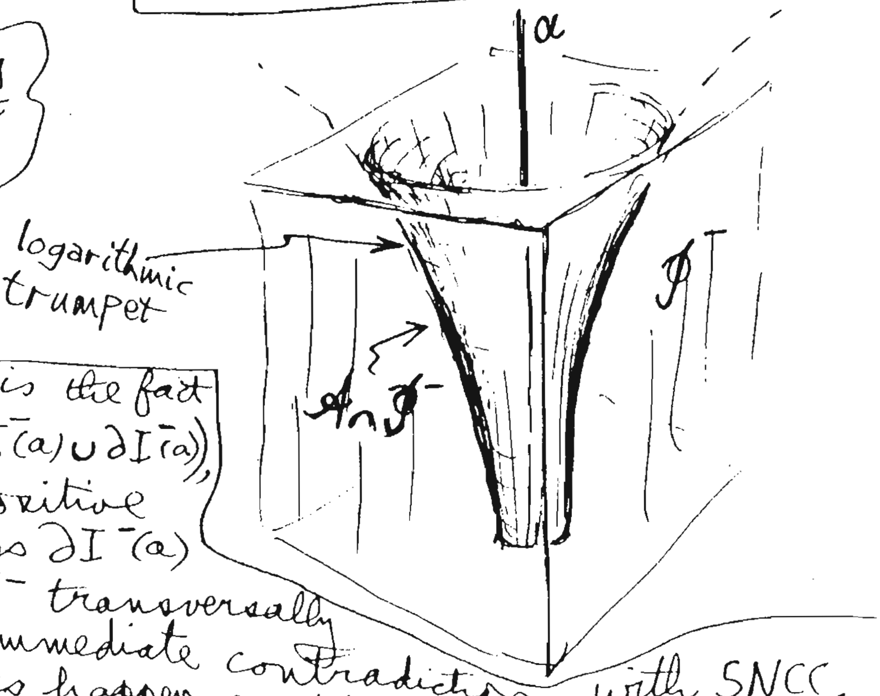


full 3-dim $\mathscr{I}^-$ picture ($\mathcal{A} \cap \mathscr{I}^-$ is logarithmic trumpet)

Now consider negative mass. The picture is like the one above but the other way up:



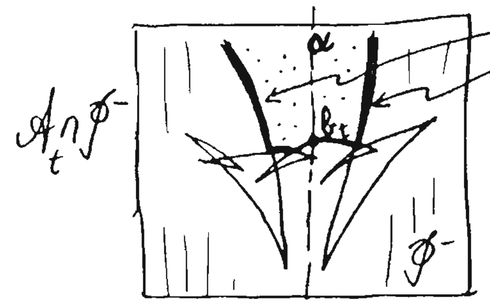




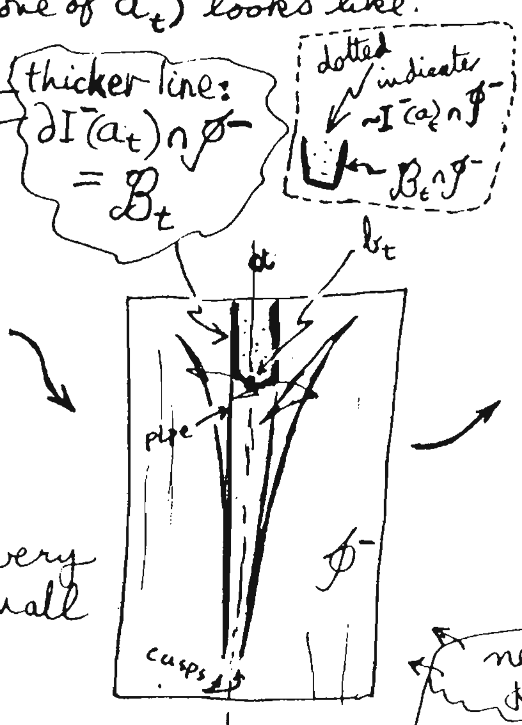
What's not so clear from this picture is the fact that $\mathscr{I}^- \subset \overline{I^-(a)}$ $(= I^-(a) \cup \partial I^-(a))$, so that, as in the positive and zero mass cases $\partial I^-(a)$ does not intersect $\mathscr{I}^-$ transversally and we have no immediate contradiction with SNCC.

To see what does happen, consider a smooth family of points a_t $(t \geq 0)$ lying on (say) a null ray terminating at a, where $a = a_0$. For each $t > 0$, $\partial I^-(a)$ intersects $\mathscr{I}$

$\mathscr{I}^-$ transversally in a (not necessarily smooth) cut of $\mathscr{I}^-$. The point $b_t = \alpha \cap \partial I^-(a_t)$ lies on a null ray on $\partial I^-(a_t)$ free of conjugate points (except possibly at end-points). As $t \to 0$, the intersection $\mathcal{A}_t \cap \mathscr{I}^-$ (where $\mathcal{A}_t$ is past light cone of a_t) looks like:



t fairly small



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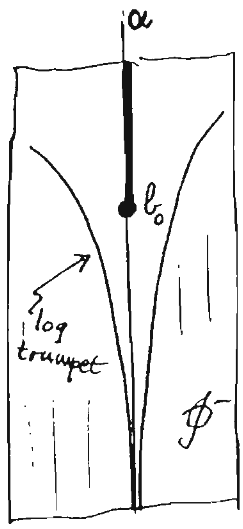
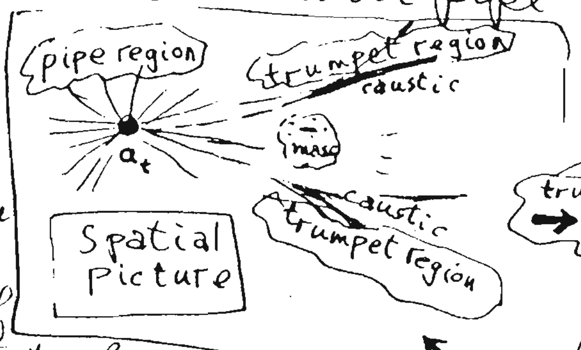


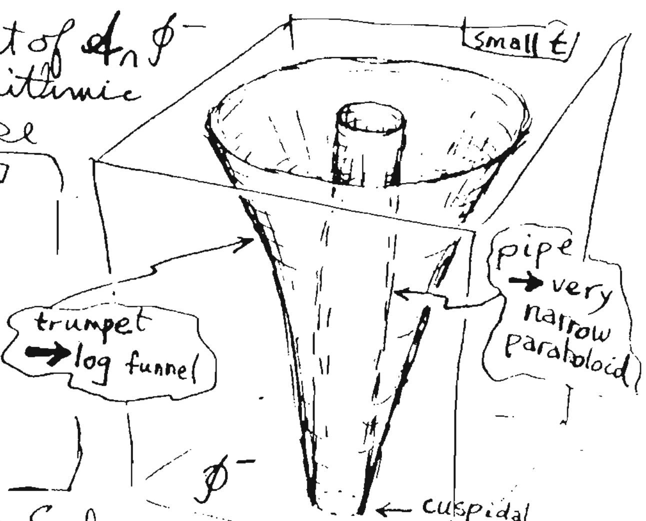
t = 0

negative mass pictures

The region $\mathcal{B}_t = \partial I^-(a_t) \cap \mathscr{I}^-$ moves continuously with t, finally converging on a future-endless segment of α with past end-point $b_0 = b$.

One must show that the part of $\mathcal{A}_t \cap \mathscr{I}^-$ that lies away from the logarithmic trumpet and the narrower pipe within it actually stabilizes as $t \to 0$ so that b_t attains a limit. The pipe and trumpet come from the resemblance of $\mathcal{M}$'s i^0 to that of a negative mass Schwarzschild solution as can be seen from examining the spatial picture. Having b_0, we find the null ray on $\partial I^-(a_0)$ from b_0 to a_0, which must be free of conjugate points: the desired contradiction with SNCC.





No doubt the argument can be strengthened in various ways (e.g. allowing conj. pts at $\mathscr{I}^\pm$ or the mass to be zero). Further details elsewhere.

~ Roger Penrose

A quasi-local mass construction with positive mass

A. Dougan and L.J. Mason

In this note we propose a pair of modifications to Penrose's quasi-local mass construction that not only lead to a definition of a real 4-momentum and mass of the gravitational and matter fields within a two surface $\mathcal{S}$, but also have the property that the momentum can be proved to be future pointing when the 2-surface can be spanned by a three surface on which the data satisfies the dominant energy condition (the proof also requires that the 2-surface be convex). The new definition reproduces the good properties of the quasi-local mass construction—it gives zero in flat space, and the correct results in linearized theory and at infinity.

Motivation: In Mason (1989) (see also Mason & Frauendiener 1990) the components of the angular momentum twistor associated to a 2-surface $\mathcal{S}$ are interpreted as the values of the Hamiltonians that generate motions of a spanning 3-surface $\mathcal{H}$ whose boundary value on $\mathcal{S}$ are 'quasi-Killing vectors' constructed out of solutions to the 2-surface twistor equations. (The value of the Hamiltonians that generate motions of $\mathcal{H}$ in space-time depends only on the boundary value of the deformation 4-vector field on $\mathcal{S}$.)

For Penrose's quasi-local mass construction (Penrose 1982) the quasi-Killing vectors are constructed out of the four linearly independent solutions of the twistor equation $\omega^A_\alpha = (\omega^A_0, \ldots, \omega^A_3)$. They are given by $K^{AA'} = K^{\alpha\beta}\omega^A_\alpha \pi^{A'}_\beta$ where $K^{\alpha\beta} = K^{(\alpha\beta)}$ is a matrix of constants and $\pi_{A'\alpha}$ are the π-parts of the ω^A_α defined by $d\omega^A|_{\mathcal{S}} = -i\pi_{A'}dx^{AA'}|_{\mathcal{S}}$. The value of the Hamiltonian that generates deformations of $\mathcal{H}$ with boundary value $K^{AA'}$ on $\mathcal{S}$ is obtained by inserting this decomposition of $K^{AA'}$ into the Witten-Nester integral:

$$H(K^{AA'}) = A_{\alpha\beta}K^{\alpha\beta} = -i\oint_{\mathcal{S}} K^{\alpha\beta}\omega^A_\alpha d\pi^{A'}_\beta \wedge dx_{AA'}.$$

This expression depends on $K^{AA'}$ *and* its decomposition into spinors. By use of $d(-i\pi_{A'}dx^{AA'}) = d^2\omega^A = R_B{}^A\omega^B$ it can be seen that this is equivalent to Penrose's original definition.

The new momentum definition: In order to define a real 4-momentum we must have a definition of real 'quasi-translations' at $\mathcal{S}$. Two definitions follow. The equation $\bar{\eth}\pi_{A'} = 0$, resp. $\eth\pi_{A'} = 0$ (where $\bar{\eth} = \bar{m}^a\nabla_a$, $\eth = m^a\nabla_a$, $\bar{m}^a = \iota^A o^{A'}$ and $m^a = o^A\iota^{A'}$ with $o^{A'}o^A$, $\iota^A\iota^{A'}$ the outward resp. inward null normal etc.) in general has just 2 linearly independent solutions on $\mathcal{S}$, since this equation can be thought of as the condition that $\pi_{A'}$ is a holomorphic (resp. anti-holomorphic) section of the spin bundle $\mathbf{S}_{A'}$ on the sphere $\mathcal{S}$ where the complex structure on $\mathcal{S}$ is that induced from the space-time metric, and that on $\mathbf{S}_{A'}$ arises from the space-time spin connection. Generically $\mathbf{S}_{A'}$ is trivial as a holomorphic vector bundle on $\mathcal{S}$ and so there exists precisely two solutions $(\pi^{0'}_{A'}, \pi^{1'}_{A'}) = \pi^{A'}_{A'}$. (This type of idea is used in KPT's 1983 definition of quasi-local charges for Yang-Mills.)

We can now define a 'quasi-translation' to be a 4-vector field on $\mathcal{S}$ of the form

$$K_{AA'} = K_{\underline{A}\underline{A}'}\pi_{A'}^{\underline{A}'}\bar{\pi}_{A}^{\underline{A}}$$

where the $K_{\underline{A}\underline{A}'}$ are constants. This can now be inserted into the Witten-Nester form to obtain the corresponding values of the momenta. The quasi-local momentum can thus be defined (modulo irrelevant overall real constants) as:

$$P^{\underline{A}\underline{A}'} = i\oint_{\mathcal{S}} \pi_{A'}^{\underline{A}'} d\bar{\pi}_{A}^{\underline{A}} \wedge dx^{AA'}$$

The mass. In order to define a mass, we must be able to define a constant $\varepsilon_{\underline{A}\underline{B}}$ so that we can define:

$$m^2 = P^{\underline{A}\underline{A}'}P^{\underline{B}\underline{B}'}\varepsilon_{\underline{A}\underline{B}}\varepsilon_{\underline{A}'\underline{B}'}.$$

The natural definition is $\varepsilon^{\underline{A}'\underline{B}'} = \pi_{A'}^{\underline{A}'}\pi_{B'}^{\underline{B}'}\varepsilon^{A'B'}$. It follows from $\bar{\eth}\pi_{A'}^{\underline{A}'} = 0$ that $\bar{\eth}\varepsilon^{\underline{A}'\underline{B}'} = 0$, so that the $\varepsilon^{\underline{A}'\underline{B}'}$ are holomorphic and global functions on the sphere and hence, by Liouville's theorem, constant.

Flat space, linearised theory and infinity. In flat space, the $\pi_{A'}$ are guaranteed to be the restriction to $\mathcal{S}$ of the constant spinors, since they certainly satisfy the equation, and the solutions are unique. The integrand therefore vanishes giving the correct answer $P_{\underline{A}\underline{A}'} = 0$. In linearized theory one can again, with a little work, see that the right answer is obtained (one needs to integrate potentially awkward terms by parts in order to see that they vanish). Asymptotically at space-like infinity, the $\pi_{A'}$'s are the asymptotically constant spinors (again because the asymptotically constant spinors satisfy $\bar{\eth}\pi_{A'} = 0$ and therefore span the solution space) and the expression reduces to the Witten-Nester expression for the ADM energy. At null infinity there is the subtlety that only one of the definitions gives the correct asymptotic spin space depending on whether one is at future or past null infinity.

Positivity. It is essential for a good definition of momentum that it should be future pointing. The following argument is an adaptation of ideas in Ludvigsen & Vickers (1983) based on Witten (1981). In the following we show that $P^{00'}$ is positive, and write, for simplicity, $\pi_{A'} = \pi_{A'}^{0'}$.

> ***Theorem.*** The quasi-local momentum $P^{00'}$ defined by the $\eth\pi_{A'} = 0$ (resp. $\bar{\eth}\pi_{A'} = 0$) is positive whenever $\rho < 0$ (resp. $\rho' > 0$).

Proof. Let $\lambda_{A'}$ be some field defined on a 2-surface $\mathcal{S}$ spanned by some non singular space-like 3-surface $\mathcal{H}$. Let $I_\lambda(\mathcal{S})$ be the integral of the Witten-Nester 2-form $\Lambda = -i\bar{\lambda}_A d\lambda_{A'} \wedge dx^{AA'}$ over $\mathcal{S}$. In spin coefficients and the GHP formalism this may be written:

$$I_\lambda(\mathcal{S}) = \oint_{\mathcal{S}} \{\bar\lambda_1(\eth\lambda_{0'} + \rho\lambda_{1'}) - \bar\lambda_0(\bar\eth\lambda_{1'} + \rho'\lambda_{0'})\}dS \qquad (1).$$

Consider first the system of equations $\eth\pi_{A'} = 0$:

$$\eth\pi_{0'} + \rho\pi_{1'} = 0, \qquad \eth\pi_{1'} + \bar\sigma'\pi_{0'} = 0 \qquad (2,3).$$

Then using (2) and integrating by parts we get:

$$I_\pi(\mathcal{S}) = -\oint_{\mathcal{S}} (\rho'\bar\pi_0\pi_{0'} + \rho\bar\pi_1\pi_{1'})dS \qquad (4)$$

Since the Sen-Witten equation on $\mathcal{H}$ consists of an elliptic system of two first order P.D.E's, we may find a solution $\tilde\pi_{A'}$ satisfying the boundary condition

$$\tilde\pi_{0'} = \pi_{0'} \qquad (5)$$

on $\mathcal{S}$. In general $\tilde\pi_{1'}$ will differ from $\pi_{1'}$ on $\mathcal{S}$. Denote this difference by:

$$Y = \tilde\pi_{1'} - \pi_{1'} \qquad (6)$$

We now relate $I_\pi(\mathcal{S})$ to $I_{\tilde\pi}(\mathcal{S})$:

$$\begin{aligned}
I_{\tilde\pi}(\mathcal{S}) &= \oint_{\mathcal{S}} \left\{\bar{\tilde\pi}_1(\eth\tilde\pi_{0'} + \rho\tilde\pi_{1'}) - \bar{\tilde\pi}_0(\bar\eth\tilde\pi_{1'} + \rho'\tilde\pi_{0'})\right\}dS \\
&= \oint_{\mathcal{S}} \left\{\bar{\tilde\pi}_1(\eth\pi_{0'} + \rho\tilde\pi_{1'}) - \bar\pi_0(\bar\eth\tilde\pi_{1'} + \rho'\pi_{0'})\right\}dS \\
&= \oint_{\mathcal{S}} \left\{\bar{\tilde\pi}_1(-\rho\pi_{1'} + \rho\tilde\pi_{1'}) - \rho\bar\pi_1\tilde\pi_{1'} - \rho'\bar\pi_0\pi_{0'})\right\}dS \\
&= \oint_{\mathcal{S}} \left\{-\rho'\bar\pi_0\pi_{0'} - \rho\bar\pi_1\pi_{1'} + \rho(\bar{\tilde\pi}_1 - \bar\pi_1)(\tilde\pi_{1'} - \pi_{1'})\right\}dS \\
&= I_\pi(\mathcal{S}) + \oint_{\mathcal{S}} \rho Y\bar Y dS
\end{aligned}$$

Where we have used equations (2), (4), (5) and (6) and an integration by parts. As is well known (Witten 1980) for matter satisfying the dominant energy condition $I_{\tilde\pi}(\mathcal{S}) \geq 0$ so that whenever $\rho \leq 0$, $I_\pi(\mathcal{S}) \geq 0$. This implies that $P^{\underline{A}\underline{A}'}$ is future pointing as required.

Considering next the equation $\bar\eth\pi_{A'} = 0$, an analogous argument to the one above but now with $\tilde\pi_{1'} = \pi_{1'}$ as boundary conditions for the Sen-Witten equation will show positivity whenever $\rho' \geq 0$.□

The conditions $\rho \leq 0$ or $\rho' \geq 0$ are the condition that the two surface is convex, i.e. that there should be no indentations. This will be satisfied by a wide class of 2-surfaces in a generic space-time.

Angular momentum: One can define more general quasi-Killing vectors using local twistors, $(\omega^A, \pi_{A'})$ restricted to $\mathcal{S}$ satifying either $\eth(\omega^A, \pi_{A'}) = 0$ or $\bar{\eth}(\omega^A, \pi_{A'}) = 0$ where $\eth$ and $\bar{\eth}$ act according to the local twistor connection. These equations are guaranteed to have just four independent solutions generically since as before these are $\bar{\eth}$ equations whose solutions are holomorphic sections of a holomorphic vector bundle on the sphere $\mathcal{S}$. Generically the holomorphic vector bundle will be trivial and so there will be just four linearly independent solutions. These can be used to define quasi-Killing vectors, and quasi-conformal Killing vectors as in Mason & Frauendiener which then give rise to 'conserved' quantities by substitution into the Witten-Nester form. (When $R_{ab} = 0$ on $\mathcal{S}$, it makes consistent sense to set $\pi_{A'} = 0$ in such a local twistor and then we can retrieve the quasi-local momentum above within the scheme.)

Many thanks to K.P.Tod for useful discussions.

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Monopoles and Yang-Baxter equations

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M. K. Murray

It has been known for some time that the Yang-Baxter equations can be solved using elliptic curves. More recently it was discovered [1, 2] that the YBE for the N state chiral Potts model could be solved using special curves of genus $(N-1)^2$.

The curves that arise can be defined as follows. Let $k^2 + k'^2 = 1$ and then intersect the following two Fermat surfaces in $\mathbb{CP}_3$:

$$a^N + k' b^N = k d^N$$
$$k' a^N + b^N = k c^N.$$

This gives a curve B_N with a high degree of symmetry. In fact $\mathbb{Z}_N^4$ acts on $\mathbb{CP}_3$ by $(\omega_1, \omega_2, \omega_3, \omega_4)[a, b, c, d] = [\omega_1 a, \omega_2 b, \omega_3 c, \omega_4 d]$ (where $\omega_i^N = 1$) and this fixes B_N. (Of course the diagonal $(\omega, \omega, \omega, \omega)$ acts trivially so it is really a $\mathbb{Z}_N^3 = \mathbb{Z}_N^4/\Delta$ action). The quotient of B_N by a free action of a $\mathbb{Z}_N$ subgroup gives the curve Σ_N of genus $(N-1)^2$.

On a visit to Canberra in 1989 the first author conjectured that these curves should be related to the spectral curves of an $SU(2)$ monopole of charge N. These are also a special class of curves of genus $(N-1)^2$. We now understand how such a relationship exists for hyperbolic monopoles with Higgs field equal to zero. Consider the $\mathbb{C}^\times$ action on $\mathbb{CP}_3$ given by

$\lambda\,[a,b,c,d] \longmapsto [\lambda a, b, c, \lambda d]$. If we remove the lines $C_1 = [0,b,c,0]$ and $C_2 = [a,0,0,d]$ of fixed points this is a free action with quotient the quadric $Q = \mathbb{P}_1 \times \mathbb{P}_1$. In fact it realizes $\mathbb{CP}_3 - C_1 \cup C_2$ as the $\mathbb{C}^\times$ bundle of the line bundle $\mathcal{O}(1,-1)$ over the quadric. We shall call this bundle L.

The important fact is that B_N intersects the orbits of the $\mathbb{C}^\times$ action in the orbits of $\mathbb{Z}_N$ considered as a subgroup inside $\mathbb{C}^\times$. So projecting to Q divides B_N by $\mathbb{Z}_N$ to give the curve Σ_N in Q. It is easy to check that Σ_N is in the linear system $\mathcal{O}(N,N)$. In fact we can say more. If we factor the $\mathbb{C}^\times$ bundle $\mathbb{CP}_3 - C_1 \cup C_2$ by $\mathbb{Z}_N$ this gives the bundle $L^N \to Q$ and the curve B_N becomes a section over Σ_N. So Σ_N satisfies the constraint

$$L^N/_{\Sigma_N} \simeq \mathcal{O}.$$

In the theory of hyperbolic monopoles [3] the monopole is determined by a spectral curve S_N in Q in the linear system $\mathcal{O}(N,N)$. This satisfies a constraint

$$L^{2p+N}/_{S_N} \simeq \mathcal{O}.$$

where p is the norm of the Higgs field at infinity.

So this shows that the curve Σ_N is that for a monopole with zero Higgs field! Strictly speaking such monopoles are trivial so we have to interpret Σ_N as arising from some limit of monopoles. Work in progress suggests that this can be done via the rational map of the monopole.

Finally notice that we can turn this discussion about and say that a curve Σ_N as above with the constraint $L^N/_{\Sigma_N} \simeq \mathcal{O}$ is equivalent to a curve in $\mathbb{C}P_3$ with no constraint except invariance under $\mathbb{Z}_N \subset \mathbb{C}^{\times}$. Looked at from this point of view Baxters curves B_N are special curves invariant under two more actions of $\mathbb{Z}_N$. The more general curves we have discussed here have moduli spaces of dimension $4N$ and it is hoped that this means that Baxter's curves can be generalised.

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H-Space—a universal integrable system?

L.J.Mason

The following speculations have not been fulfilled yet (and may never) but I feel that the concrete aspects of the ideas are of interest and the various relations involved are intriguing.

Motivation. There is a large forest of integrable systems. Richard Ward, amongst others, has pointed out that many, if not indeed most integrable systems are reductions of the self-dual Yang-Mills equations. This observation isn't just a question of bookkeeping, it gives a substantial insight into the theory underlying these equations as the inverse scattering transform for these systems can be understood as a symmetry reduction of the Ward construction for solutions of the self-dual Yang-Mills equations (Woodhouse & Mason 1988 and Mason & Sparling 1989 & preprint, the symmetry reduction can, however, be somewhat nontrivial—see in particular Woodhouse & Mason in which non-Hausdorff Riemann surfaces play an essential role).

Two gaps in the story are as follows. Firstly that there appears to be genuine difficulties to incorporating the KP and Davey-Stewartson equations. There is little difficulty in incorporating integrable systems into some kind of twistor framework if the inverse scattering transform is realised by means of the solution of a Riemann-Hilbert problem. However the inverse scattering problem for the KP equations is more subtle and requires the solution of a 'non-local Riemann-Hilbert problem'. This gap is particularly irritating in view of the theoretical importance that the KP equations have acquired with its relations to the theory of Riemann surfaces etc. The second gap is that there appears to be little rôle for the self-dual vacuum equations and its twistor construction, RP's nonlinear graviton construction—this, it should be pointed out, is not based on the solution of a Riemann-Hilbert problem either. However I should like to make the following conjecture:

Conjecture. The KP and Davey-Stewartson equations are reductions of the self-dual Einstein equations.

The circumstantial evidence is as follows. (The self-duality equations are taken to be concerned with space-times with metric of signature (2,2).)

Lemma 1. KP can be obtained in the limit as $n \to \infty$ of the $SL(n)$ self-dual Yang-Mills equations reduced by two orthogonal null translations.
(This extends the results of Mason & Sparling 1989.)

Lemma 2. (Hoppe, J.) The Lie algebra of the area preserving diffeomorphism group of a surface Σ^2, $SDiff(\Sigma^2)$ can be approximated arbitrarily closely by that of $SL(n)$ as $n \to \infty$.

Lemma 3. The self-dual Einstein equations are equivalent to the self-dual Yang-Mills equations reduced by two orthogonal null translations with gauge group $SDiff(\Sigma^2)$. (This extends the results of Mason & Newman 1989)□

Remark. If it were the case that $SL(n)$ were a subgroup of $SL(\infty) = SDiff(\Sigma^2)$ then these results would imply that all 2-dimensional integrable models obtainable as reductions from the self-dual Yang-Mills equations (at least by translations). Hence the title of this note and the question mark. However, my current opinion is that $SL(n)$ is only a subgroup of $SDiff(\Sigma^2)$ for $n = 2$. This still yields a reasonable class of integrable systems and certainly the more famous ones such as the KdV, nonlinear Schrodinger and the sine-Gordon equations.

Proof of lemma 1. I shall use the presentation of the KP hierarchy due to Gelfand and Dicke. See for instance Segal & Wilson in the proceedings of the I.H.E.S for a description of these ideas. The equations of the KP hierarchy are the consistency conditions for the existence of a solution ψ to the following system of linear partial differential equations

$$(\partial_{t_2} - (Q^2)_+)\psi = 0,\ (\partial_{t_3} - (Q^3)_+)\psi = 0,\ \cdots,\ (\partial_{t_r} - (Q^r)_+)\psi = 0,\ \cdots$$

where $(Q^r)_+$ is an r^{th} order O.D.E. in the x variable, $Q_r = (\partial_x)^r + ru(\partial_x)^{r-2} + \cdots + w_r$ and $u(x, t_2, t_3, \cdots)$ is the subject of the KP hierarchy equation and w_r is some function which will be determined in terms of u by the equations. The notation is intended to indicate that the ordinary differential operators $(Q^r)_+$ are the differential operator part of the pseudo-differential operator Q raised to the r^{th} power where $Q = \partial_x + u(\partial_x)^{-1} +$ (lower order) and where $(\partial_x)^{-1}$ is a formal pseudo-differential operator defined by the relation $(\partial_x)^{-1} f = f(\partial_x)^{-1} + \sum_{i=1}^{\infty} (-\partial_x)^i f (\partial_x)^{-i-1}$.

The basic KP equation is the equation on $u(x, t_1, t_2)$ that follows from the consistency conditions for $(\partial_{t_2} - (Q^2)_+)\psi = 0$ and $(\partial_{t_3} - (Q^3)_+)\psi = 0$ alone. The evolution in the higher time variables are symmetries of the basic equations (and each other). If one imposes invariance in the n^{th} time variable t_n, then the reduced system is referred to as the n^{th} generalized KdV hierarchy ($n = 2$ gives the standard KdV hierarchy and $n = 3$ the Boussinesq).

The basic idea is that the operators $(Q^r)_+$ can be thought of as infinite dimensional matrices acting on $L^2(\mathbf{R})$ where x is a coordinate on $\mathbf{R}$. One can approximate this by $n \times n$ matrices by imposing a symmetry in the n^{th} time variable since then (Fourier transforming ψ in the t_n variable) we have $(Q^n)_+ \psi = \lambda\psi$ and we consider only ψ lying in the n-dimensional solution space of this equation, represented, say, by ψ and its first $(n-1)$-derivatives with respect to x. With this reduction we have:

$$(\partial_{t_2} - (Q^2)_+)\psi = 0 \text{ reduces to } \left\{ \partial_{t_2} - \begin{pmatrix} 2u & 0 & 1 \cdots & 0_{n-3} \\ \cdot & 2u & \ddots & 1 \\ \cdot & \cdot & \ddots & 0 \\ \cdot & \cdot & \cdot & (2-n)u \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 1 & 0 & \ddots & 0 \\ 0 & 1 & \cdots & 0 \end{pmatrix} \right\} \underline{\psi} = 0$$

$$(\partial_{t_3} - (Q^3)_+)\psi = 0 \text{ reduces to } \left\{ \partial_{t_2} - \begin{pmatrix} u_x & u & 0 & 1 & \ddots & 0_{n-4} \\ \cdot & \cdot & u_x & u & \ddots & 1 \\ \cdot & \cdot & \cdot & \cdot & \ddots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ddots \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & 1 \cdots & 0 \end{pmatrix} \right\} \underline{\psi} = 0$$

where 0_r is the $r \times r$ zero matrix. This matrix linear system is linear in the spectral parameter λ and can be seen to be the linear system of a reduction of SDYM with 2 null orthogonal translation symmetries.□

Note: A large gap in the above discussion is that the linear system is shown to be contained within the SDYM linear systems, but I have not characterised those SDYM solutions with the 2 orthogonal null symmetries that give rise to the n^{th} KdV system.

Proof of Lemma 2. These ideas are standard. One presents the Lie algebra of the area preserving diffeomorphisms of a torus by using the area form as a symplectic form and representing vector fields corresponding to elements of $LieSDiff(\Sigma^2)$ by their Hamiltonians. Let θ_1 and θ_2 be angular coordinates on the torus such that the area form is $d\theta_1 \wedge d\theta_2$, then a basis for the Hamiltonians is $H_{\underline{A}} = exp\{2\pi i(A_1\theta_1 + A_2\theta_2)\}$ where $\underline{A} = (A_1, A_2) \in \mathbb{Z} \times \mathbb{Z}$. The Lie bracket is the Poisson bracket:

$$\{H_{\underline{A}}, H_{\underline{B}}\} = (\underline{A} \wedge \underline{B}) H_{\underline{A}+\underline{B}} \quad \text{where} \quad (\underline{A} \wedge \underline{B}) = A_1 B_2 - A_2 B_1.$$

For $SL(N)$ we use a basis for the Lie algebra constructed using a pair of matrices U, V satisfying the quantum plane relations: $UV = \zeta VU$ where $\zeta^N = 1$. An explicit representation has U diagonal with powers of ζ down the diagonal $U_{ij} = \zeta^i \delta_{ij}$ and V a shift matrix $V_{ij} = \delta_{i(j+1 \ mod \ N)}$.

A basis for the Lie algebra of $SL(N)$ is then furnished by

$$T_{\underline{A}} = N \zeta^{\frac{A_1 A_2}{2}} U^{A_1} V^{A_2}.$$

The commutators are then given by:

$$[T_{\underline{A}}, T_{\underline{B}}] = N \sin \frac{2\pi \underline{A} \wedge \underline{B}}{N} T_{\underline{A}+\underline{B}} \quad \xrightarrow[N \to \infty]{} \quad (\underline{A} \wedge \underline{B}) T_{\underline{A}+\underline{B}}$$

which gives the same commutation relations as above for $H_{\underline{A}}$ in the limit as $N \to \infty$.□

Proof of lemma 3. This is, to a certain extent, a corollorary of the results in Mason & Newman (1989). In that paper it was shown that if you take the algebraic relations obtained by imposing four translational symmetries on the self-dual Yang-Mills equations and take the gauge group to be the group of volume preserving diffeomorphisms of some 4-manifold then, roughly speaking, one obtains the self-dual vacuum equations. Lemma 3 can be reformulated so as to be a special case of this.

The self-dual Yang-Mills equations on $\mathbf{R}^4$ with metric $ds^2 = dudy + dvdx$ (signature 2,2) are the integrability conditions on connection components (A_u, A_v, A_x, A_y) in the Lie algebra of the gauge group for the the linear system

$$\{\partial_u + A_u + \lambda(\partial_x + A_x)\}\psi = 0 \qquad \{\partial_v + A_v + \lambda(\partial_y + A_y)\}\psi = 0.$$

When G is $SDiff(\Sigma^2)$ the connection components are all vector fields on Σ^2 (depending also on the coordinates on $\mathbf{R}^4$). Impose two translational symmetries on the $\mathbf{R}^4$ so that the connection components depend only on the quotient variables on $\mathbf{R}^2$. The linear system then reduces to the system $\{V_u + \lambda V_x\}\psi = 0 = \{V_v + \lambda V_y\}\psi$ where the V's are vector fields on $\mathbf{R}^2 \times \Sigma^2$. These vector fields preserve the natural volume form on $\mathbf{R}^2 \times \Sigma^2$ and so determine elements of the Lie algebra of the volume preserving diffeomorphism group. The linear system is precisely that for the self-dual Yang-Mills equations with 4 translational symmetries and gauge group the volume preserving diffeomorphisms of $\mathbf{R}^2 \times \Sigma^2$.

Concretely introduce coordinates (p, q) on Σ^2 so that the area form is the symplectic form $dp \wedge dq$, and suppose the symmetries to be in the x and y directions so that the variables depend only on the coordinates (u, v) on $\mathbf{R}^2$. Represent the vector fields A_x on Σ^2 by their Hamiltonians denoted h_x etc.. The field equations are

$$[\partial_u + A_u + \lambda A_x, \partial_v + A_v + \lambda A_y] = 0$$

The first implication of this is that $\lambda^2[A_x, A_y] = 0$ so that we can choose coordinates on Σ^2 so that $A_x = \partial_q$ and $A_y = \partial_p$. The term proportional to λ implies $\partial_q h_v = \partial_p h_u$, so that $h_v = \partial_p g$ and $h_u = \partial_q g$ for some $g \equiv g(u, v, q, p)$. The final equation yields in terms of g

$$\partial_u \partial_p g - \partial_v \partial_q g + (\partial_p^2 g)(\partial_q^2 g) - (\partial_q \partial_q g)^2 = 0$$

which is Plebanski's second heavenly equation.□

Thanks to George Sparling for discussions.

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Some Quaternionically Equivalent Einstein Metrics

Andrew Swann

If M is a 4-manifold, not necessarily compact, admitting two Einstein metrics g_1, g_2 in the same conformal class such that the scalar curvature κ_1 of g_1 is non-zero while $\kappa_2 = 0$, then Brinkman [1] showed that M is conformally flat. This result may be restated quaternionically. By a quaternionic structure on a $4n$-manifold we mean a reduction of the structure group to $GL(n,\mathbb{H}) \times_{\mathbb{Z}/2} GL(1,\mathbb{H})$. This is equivalent to requiring that M has a rank three subbundle $\mathcal{G} \subset \operatorname{End} TM$ which locally has a basis I, J, K satisfying

$$I^2 = J^2 = -1 \quad \text{and} \quad IJ = K = -JI. \tag{$*$}$$

Now $GL(1,\mathbb{H})$ is isomorphic to $SU(2) \times \mathbb{R}_{>0}$, so when $n = 1$ we obtain the conformal group $CO(4) \cong GL(1,\mathbb{H}) \times_{\mathbb{Z}/2} GL(1,\mathbb{H})$ and Brinkman's result tells us about Einstein metrics with the same quaternionic structure.

If M^{4n}, $n \geqslant 2$, has a quaternionic structure and a compatible metric g, we may embed $\mathcal{G}$ in $\Lambda^2 T^*M$ by $I \mapsto \omega_I$, where $\omega_I(X,Y) = g(X, IY)$, and define a global 4-form Ω via the local formula $\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$. If Ω is parallel with respect to the Levi-Civita connection, M is said to be quaternionic Kähler. The fundamental example of such a manifold is quaternionic projective space $\mathbb{HP}(n)$ with its usual symmetric metric. Alekseevskiĭ [2] showed that quaternionic Kähler metrics are automatically Einstein and that the curvature tensor may be decomposed as

$$R = \lambda R_{\mathbb{HP}(n)} + R_0,$$

where λ is a constant positive multiple of the scalar curvature and $R_{\mathbb{HP}(n)}$ and R_0 have the symmetries of the curvature tensors of $\mathbb{HP}(n)$ and a hyperKähler metric, respectively. (For a hyperKähler manifold, $\mathcal{G}$ is trivialized by parallel complex structures satisfying ($*$).) If E and H are bundles associated to the basic representations of $Sp(n)$ and $Sp(1) \cong SU(2)$ on $\mathbb{C}^{2n}$ and $\mathbb{C}^2$, respectively, then $T_{\mathbb{C}}M \cong E \otimes_{\mathbb{C}} H$, $\mathcal{G}$ is the second symmetric power S^2H and $R_0 \in S^4E$.

Now suppose M^{4n} admits two metrics g^h, g^q with the same quaternionic structure such that g^h is hyperKähler (and hence scalar flat) and g^q is quaternionic Kähler with non-zero scalar curvature. Let ∇^h and ∇^q be the Levi-Civita connections of these metrics. The hyperKähler structure trivialises H and we obtain a section $h \in \Gamma(H)$ with $\nabla^h h = 0$. Since the twistor operator $D\colon H \xrightarrow{\nabla} H \otimes T^* \cong E \oplus (E \otimes S^2H) \to E \otimes S^2H$ is quaternionically invariant [3], $\nabla^q h = e$ is a section of E. Let $\tilde{h} = jh$, $\tilde{e} = je$ and consider the vector field $X = e\tilde{h} - \tilde{e}h$. Now $T \otimes T^*$ decomposes as

$$S^2H + S^2E + \mathsf{R} + \Lambda_0^2 E + S^2H\Lambda_0^2E + S^2HS^2E$$

and vector fields whose covariant derivatives lie in the first two terms are Killing, whilst those with derivatives in the first four summands give rise to quaternionic transformations. A simple calculation shows that X is quaternionic, while $i(e\tilde{h} + \tilde{e}h)$, $eh + \tilde{e}\tilde{h}$ and $i(eh - \tilde{e}\tilde{h})$ are Killing vectors. In the hyperKähler structure these are just IX, JX and KX and together with X they define a local action of the group H^* in which the compact subgroup $Sp(1) \cong SU(2)$ acts isometrically, but permutes I, J and K. A Weitzenböck argument now shows that R_0 lies in $S^4E^\perp$, where $E^\perp$ is the orthogonal complement in E to the span of e and $\tilde{e}$; so the orbits of the H^*-action are flat in the hyperKähler structure. Since the vector fields span a quaternionic subspace, an argument of Gray [4] also shows that the 4-dimensional orbits are totally geodesic (with respect to either metric).

Given a hyperKähler $(4n+4)$-manifold N admitting such an H^*-action which is free, we can construct a quaternionic Kähler manifold M^{4n} as follows. Fix a complex structure I and let $U(1)$ be the subgroup of $Sp(1)$ preserving I (and hence permuting J and K). Let $\mu\colon N \to \mathfrak{u}(1)^* \cong \mathsf{R}$ be a Kähler moment map for this action. The level sets of μ are actually $Sp(1)$-invariant and $M = \mu^{-1}(x)/\,Sp(1)$ is a quaternionic Kähler manifold [5]. Letting H^* act diagonally on $N \times \mathsf{H}$ gives a quaternionic Kähler metric on N in the same quaternionic class as the original hyperKähler metric. One may construct examples of such manifolds N as bundles over quaternionic Kähler manifolds, obtaining explicit formulae for both metrics. These constructions generalise the fibration $\mathsf{H}^{n+1} \setminus \{0\} \to \mathsf{HP}(n)$. In this flat case the quaternionic Kähler metric obtained on the total space is induced by the inclusion $\mathsf{H}^{n+1} \hookrightarrow \mathsf{HP}(n+1)$.

Kronheimer [6] shows that every adjoint orbit of nilpotent elements in a complex semi-simple Lie algebra $\mathfrak{g}^{\mathbf{C}}$ has a hyperKähler metric. One may check that this structure admits an H^*-action of the type described above. If $\mathfrak{g}$ is simple, the smallest orbit fibres over a compact homogeneous quaternionic Kähler manifold and the classification in [2] shows that all such quaternionic Kähler metrics arise this way. When $\mathfrak{g} = \mathfrak{su}(3)$, the quaternionic Kähler manifold is $\mathsf{CP}(2)$ and locally one has a non-flat hyperKähler structure on the negative spin bundle (away from the zero section).

The moment map μ is actually a hyperKähler potential, that is μ is simultaneously a Kähler potential for each of the complex structures on N. A hyperKähler manifold N admits such a function if and only if it admits an H^*-action of the type described above. Also, μ is a hyperKähler potential for N if and only if the hyperKähler metric is given by

$$\nabla^2\mu = g^h.$$

Further details will appear elsewhere.

Acknowledgements. I would particularly like to thank Simon Salamon for conveying to me ideas of C.R. LeBrun and Y.S. Poon that are implicit in this note.

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A Non-Hausdorff Mini-twistor Space

This note is about another example of a non-Hausdorff complex manifold arising naturally in twistor theory. A mini-twistor space $\mathbb{T}$ is the 4-real-dimensional space of directed geodesics of a 3-real-dimensional Weyl space $\mathbb{W}$, which becomes a 2-complex-dimensional manifold if the Weyl space satisfies the Einstein-Weyl condition. Since it is defined as a space of geodesics, and geodesics can wind around in funny ways, a mini-twistor space is always liable to be non-Hausdorff. I will describe an example of a particularly simple Einstein-Weyl space where the mini-twistor space can be seen to be non-Hausdorff in a fairly tame way.

Recall first that a Weyl space $\mathbb{W}$ is a manifold with a symmetric connection D and a conformal metric [g] which is preserved by D. Given a choice g_{ab} of representative metric, the compatibility between conformal metric and conection means that we can define D in terms of the metric connection and a 1-form ω_a. Under change-of-choice of representative metric we have

$$g_{ab} \to \Omega^2 g_{ab} \; ; \quad \omega_a \to \omega_a + 2\nabla_a \log\Omega \qquad (1)$$

so that we can think of a Weyl space as the pair (g_{ab}, ω_a) subject to (1). (For more details see e.g. [H],[JT],[PT]).

The connection D has a Riemann tensor and a Ricci tensor, but the Ricci tensor is not necessarily symmetric. The Einstein-Weyl condition on $\mathbb{W}$ is that the symmetrised Ricci tensor be proportional to the (conformal) metric. This can be written out as an equation on the Ricci tensor of the representative metric and the 1-form ω_a. In 3 dimensions the equation is

$$R_{ab} - \tfrac{1}{2}\nabla_{(a}\omega_{b)} - \tfrac{1}{4}\omega_a\omega_b = \Lambda g_{ab}, \text{ some } \Lambda. \qquad (2)$$

This equation is, from its definition, conformally invariant and can be regarded as a conformally-invariant generalisation of the Einstein equations. Note that spaces conformal to Einstein spaces satisfy (2) since we can use (1) to eliminate ω_a. These examples can be recognised by the fact that ω_a is exact.

The example I want to consider comes about by conformally rescaling and making identifications on flat space. Take the metric and 1-form as

$$g = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) ; \quad \omega = 0 \qquad (3)$$

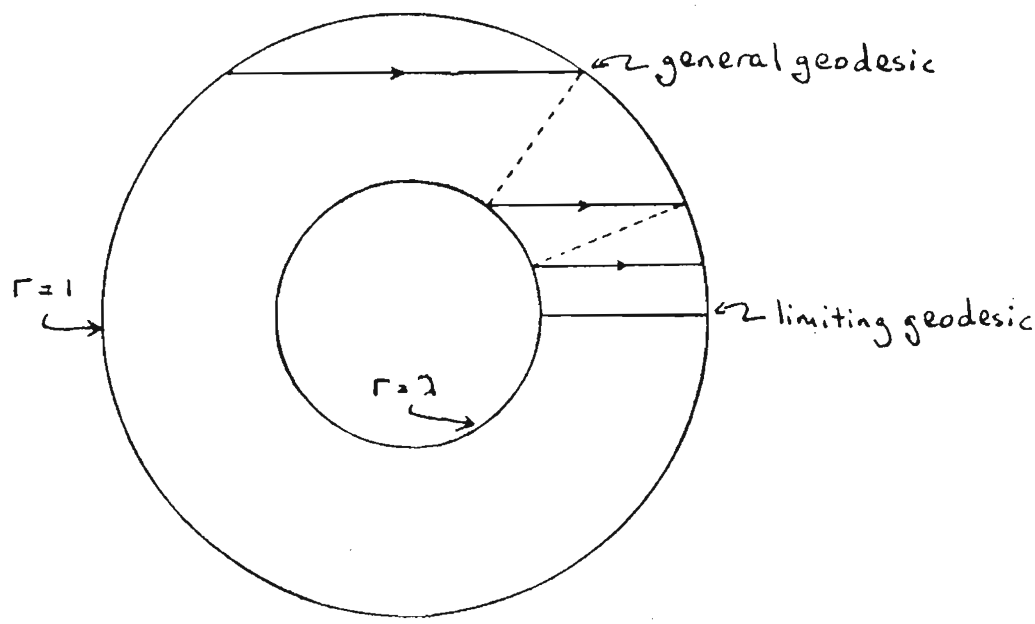
and conformally rescale with $\Omega = \exp(-\chi)$, defining $\chi = \log r$:

$$g = d\chi^2 + d\theta^2 + \sin^2\theta \, d\phi^2; \quad \omega = -2d\chi. \qquad (4)$$

Now impose a periodicity in χ to obtain an Einstein-Weyl structure on $S^1 \times S^2$ (this example is given in [PT]; part of the interest of it is that this manifold has no Einstein metric). The periodicity in χ corresponds to identifying the radial coordinate r with λr for some λ with $0<\lambda<1$.

As I said at the beginning, the space of directed geodesics of a 3-dimensional Einstein-Weyl space $\mathbf{M}$ is a 2-dimensional complex manifold $\mathbf{Z}$, the mini-twistor space of $\mathbf{M}$. For flat space, the mini-twistor space is the space of directed lines in $\mathbf{R}^3$ which can be thought of as pairs of 3-dimensional real vectors (a, b) where a is unit and b is orthogonal to a. Equivalently, this is TP1, the tangent bundle of the complex projective line. For the example to be considered here we shall need to modify this a little.

A geodesic in the $S^1 \times S^2$ Einstein-Weyl structure is as shown below:



It is basically a straight-line which, when it hits the outer sphere at r=1 is brought back to the inner sphere at $r=\lambda$ making the same angle with the radius vector. This means that in the future, the geodesic tends to a limiting one which is radially outwards and closed, while in the past it tends to a limiting one which is radially inwards and closed. In particular, this means that there are 'shadows' in the space: given a point p, points on the diameter through p but on the other side cannot be reached by geodesics through p (I am grateful to Paul Gauduchon for the suggestion that there might be shadows in this example). We shall return to these shadows below.

To construct the mini-twistor space **X**, consider first the closed radial geodesics. These correspond to the zero-section of **TP1**, ie to lines in $\mathbb{R}^3$ defined by pairs of the form (a,0), but there are 2 closed radial geodesics for each radial geodesic in flat-space so we need to double the zero-section. Next the non-radial geodesics: think of a line in flat-space as a pair (a,b) then the process of bringing this back from the outer sphere to the inner sphere in the figure above corresponds to leaving a alone but rescaling b, $b \to \lambda b$, with λ as before.

This is then the mini-twistor space: delete the zero-section from TP1; identify b with λb in the fibres; then put two copies of the zero-section back. It is non-Hausdorff at the radial geodesics ie at the doubled-up zero-section, since any geodesic which is 'near to' a radially ingoing one is also 'near to' the continuation of it to the other side as a radially outgoing one.

A point p in the Einstein-Weyl space is represented by a holomorphic curve (a 'twistor line') in the mini-twistor space. The specification of this twistor line includes, at some stage, a choice of which of a pair of doubled-up points to take. Then any twistor line through the other of the relevant pair of doubled-up points in the mini-twistor space will correspond to a point of the Einstein-Weyl space in the 'shadow' of p.

A more complicated example of a non-Hausdorff mini-twistor space should be provided by the 'Berger sphere' Einstein-Weyl space, [JT],[HT]. This corresponds to a left-invariant metric on the 3-sphere. There is a special set of geodesics like the radial ones in the example above with the property that any other geodesic tends to one of them in the future and another in the past. The mini-twistor space seems to be a sort of deformed quadric with non-Hausdorff-ness along two generators of the same family. Henrik Pedersen and I have a description of it as a 'weighted projective space' but it is a little obscure.

(Like my other TN article, the work for this was done during a most pleasant visit to Henrik Pedersen in Odense, and I gratefully acknowledge hospitality received.)

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KPT

More on harmonic morphisms

I wish to correct two errors in my last Twistor Newsletter article, to make some observations which might make that article clearer, and to describe what I think is a new way of looking at the Kerr theorem appropriate to Riemannian twistor theory (though in this last connection see Hughston and Mason CQG 5 (1988) 275).

The two errors are as follows: firstly, the generator of the congruence should have been

$$L = \frac{1-\alpha\bar{\alpha}}{1+\alpha\bar{\alpha}}\frac{\partial}{\partial z} + \frac{\alpha+\bar{\alpha}}{1+\alpha\bar{\alpha}}\frac{\partial}{\partial x} - \frac{i(\alpha-\bar{\alpha})}{1+\alpha\bar{\alpha}}\frac{\partial}{\partial y} \tag{1}$$

(this expression had some bars missing before)
and secondly $\alpha(x,y,z)$ is actually given implicitly by

$$f(x(1-\alpha^2) + iy(1+\alpha^2) - 2\alpha z, \alpha) = 0 \tag{2}$$

(this expression had some signs muddled before).

In the interest of clarity, I should also have added that:

- if $\alpha = u+iv$ then
$$\nabla^2 u = \nabla^2 v = 0 = \nabla u \cdot \nabla v \;;\; |\nabla u|^2 = |\nabla v|^2$$
ie the complex function $\alpha(x,y,z)$ obtained from (2) defines the harmonic morphism,
- $\alpha(x,y,z)$ is constant along L as given by (1),
- for a given constant value of α, the real and imaginary parts of (2) each define a plane and L is then tangent to the line of intersection of the two planes.

There is an 'elementary' interpretation of (2) as follows: a line in R^3 is given by an equation of the form

$$\mathbf{r}\wedge\mathbf{a} = \mathbf{b} \tag{3}$$

where $\mathbf{a}$ is a unit vector and $\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Parametrise the unit vectors with the complex number α according to (1), then (3) can be written

$$x(1-\alpha^2) + iy(1+\alpha^2) - 2\alpha z = \beta \tag{4}$$

where β parametrises the Argand plane orthogonal to $\mathbf{a}$. Now (2) is equivalent to taking β to be an arbitrary holomorphic function of α ie the 'intercept' is an arbitrary holomorphic function of the 'direction', both understood as complex variables. When this function is linear, the congruence is a 3-dimensional picture of the Kerr congruence and is sketched in Baird and Wood *Math. Ann. 280 (1988) 579-603.*

'Harmonic morphisms' is also the answer to the question 'what do you get from the Kerr theorem in Riemannian twistor theory (when there are no shear-free geodesic congruences)?' To see this, take a homogeneous holomorphic twistor function $F(Z^\alpha)$ and intersect the zero-set of F with a line in twistor space which is 'real' in the sense appropriate to Riemannian twistor theory. This gives

$$F(a+b\zeta, -\bar{b}+\bar{a}\zeta, 1, \zeta) = 0 \qquad (5)$$

writing $(1,\zeta)$ for the π-spinor (rather than $(1,\alpha)$ which I used at the beginning). Here a and b are complex coordinates on $\mathbf{R}^4$ and the metric is

$$ds^2 = da\,d\bar{a} + db\,d\bar{b} \qquad (6).$$

Solving (5) gives a function $\zeta(a,b,\bar{a},\bar{b})$ with

$$\frac{\partial\zeta}{\partial\bar{a}} + \zeta\frac{\partial\zeta}{\partial\bar{b}} = 0; \quad \frac{\partial\zeta}{\partial b} - \zeta\frac{\partial\zeta}{\partial a} = 0 \qquad (7)$$

from which it follows that ζ has vanishing Laplacian and null gradient in the metric (6):

$$\frac{\partial^2\zeta}{\partial a\partial\bar{a}} + \frac{\partial^2\zeta}{\partial b\partial\bar{b}} = 0; \quad \frac{\partial\zeta\partial\zeta}{\partial a\partial\bar{a}} + \frac{\partial\zeta\partial\zeta}{\partial b\partial\bar{b}} = 0 \qquad (8)$$

If we think of ζ as the stereographic coordinate on the sphere, then (8) is easily seen to be the conditions for (5) to define a harmonic morphism from $\mathbf{R}^4$ to S^2. However, (7) is stronger in that it implies that, as well as defining a harmonic morphism, ζ is constant on flat 2-planes. Note that if ζ satisfies (8) then so does any holomorphic function of ζ. In this sense, ζ defines a family of holomorphically-related harmonic morphisms constant on flat 2-planes, one of which satisfies (7).

For the converse suppose that η satisfies (8) and is constant on flat 2-planes. Define ζ by

$$\frac{\partial\eta}{\partial\bar{a}} + \zeta\frac{\partial\eta}{\partial\bar{b}} = 0 \quad \text{so that also} \quad \frac{\partial\eta}{\partial b} - \zeta\frac{\partial\eta}{\partial a} = 0$$

then it follows from the conditions on η that ζ is a holomorphic function of η and so in turn satisfies (7) and (8).

To summarise, a family of holomorphically-related harmonic morphisms from $\mathbf{R}^4$ to S^2 which are constant on 2-planes defines and is defined by a holomorphic function in twistor space. This can be called 'the Riemannian Kerr theorem'.

(This view of the Kerr theorem arose in discussions with Henrik Pedersen.)

KPT

An algorithm for the Penrose Transform

There is an algorithm for computing the E_1-term of the Penrose transform for homogeneous bundles between suitable domains in complex homogeneous spaces (see [1] for what this means) which is surprisingly simple, related to the topology of the homogeneous spaces and to Hecke modules. The significance is still a little unclear, but the hope is that the algorithm will allow us to establish precisely what maps can occur as higher differentials in the remainder of the hypercohomology spectral sequence. We illustrate for the ordinary twistor space $\leftrightarrow$ Minkowski space correspondence.

FACT 1 $\mathbb{PT}$ & $\mathbb{M}$ have certain diagrams attached which (1) record the topology of the spaces (by Morse strata) (2) record the possible (regular) homogeneous bundles on these spaces.

$$\mathbb{PT}: \quad \mathcal{O}_{\mathbb{P}} \xrightarrow{1} \Omega^1_{\mathbb{P}} \xrightarrow{2} \Omega^2_{\mathbb{P}} \xrightarrow{3} \Omega^3_{\mathbb{P}}$$

$$\mathbb{M}: \quad \begin{tikzcd} & & \Omega^2_+ \arrow[dr, "1"] & & \\ \mathcal{O} \arrow[r, "2"] & \Omega^1 \arrow[ur, "3"] \arrow[dr, "1"] & & \Omega^3 \arrow[r, "2"] & \Omega^4 \\ & & \Omega^2_- \arrow[ur, "3"] & & \end{tikzcd}$$

HASSE diagrams with trivial ∞-char bundles

FACT 2 The E_1 term for $\Omega^3_{\mathbb{P}}$ is
$$\begin{array}{|ccc} \Omega^2_+ & \Omega^3 & \Omega^4 \\ 0 & 0 & 0 \\ \hline \end{array}$$

(the E_1 term for the extreme rhs space in each Penrose transform is easy to compute – it's always a row, as high as possible)

FACT 3 Define a polynomial $E_X = \sum E_1^{p,q} u^p v^q$ for $E_1^{p,q}$ the first term in the Penrose transform of X on $\mathbb{PT}$ (so $X = \Omega^2_{\mathbb{P}}$ eg.) This is just a way of keeping track of the terms in E_1. If Y is a bundle in the diagram for $\mathbb{PT}$ or $\mathbb{M}$ define

(i) $T_i Y = v^{-1} Y$ if edge i is not incident with Y

(ii) $T_i Y = Z$ if $Z \xrightarrow{i} Y$

Then any bundle on $\mathbb{PT}$ is got by applying successive T_i to $\Omega^3_{\mathbb{P}}$ & we have, if $Z \xrightarrow{i} Y$: $\boxed{E_Z = T_i E_Y}$

Example $E_{\Omega^2_{\mathbb{P}}} = T_3 E_{\Omega^3_{\mathbb{P}}}$: $E'_{\Omega^2_{\mathbb{P}}} = \begin{array}{|ccc} \Omega^1 & \Omega^2_- & 0 \\ 0 & 0 & \Omega^4 \\ \hline \end{array}$

$E'_{\Omega^1_{\mathbb{P}}} = T_2 E'_{\Omega^2_{\mathbb{P}}} = \begin{array}{|ccc} \mathcal{O} & 0 & 0 \\ 0 & \Omega^2_- & \Omega^3 \\ \hline \end{array}$ $\qquad E'_{\mathcal{O}_{\mathbb{P}}} = T_1 E'_{\Omega^1_{\mathbb{P}}} = \begin{array}{|ccc} \mathcal{O} & \Omega^1 & \Omega^2_+ \\ \hline \end{array}$

Remarks i) T_i generate a Hecke algebra (deformation of group algebra of S_4); such play a key role in the famous Kazhdan–Lusztig conjecture.
ii) The recipe works for all groups/parabolics & gives an (Euler) character for all the cohomology groups which is easily computable. This is interesting in itself to representation theorists.

Rob Baston

[1] Baston/Eastwood: "The Penrose Transform" OUP. 1989.

The Penrose Transform without spectral sequences.

On twistor space there are three short exact sequences connecting local twistor bundles and the differential forms. Thus

$$\begin{array}{c} 0 \to \Omega^3 \to \mathcal{T}^\alpha(-3) \to \Omega^2 \to 0 \\ 0 \to \Omega^2 \to \mathcal{T}^{[\alpha\beta]}(-2) \to \Omega^1 \to 0 \\ 0 \to \Omega^1 \to \mathcal{T}_\alpha(-1) \to \mathcal{O} \to 0. \end{array} \tag{1}$$

The maps are inherited from the *Koszul complex* for twistor space:

$$0 \to T^\alpha \to T^{[\alpha\beta]} \to T_\alpha \to \mathbf{C} \to 0,$$

and are given by $\omega \to Z^\alpha\omega$, $P^\alpha \to P^{[\alpha}Z^{\beta]}$, $Q^{[\alpha\beta]} \to \epsilon_{\alpha\beta\gamma\delta}Q^{\beta\gamma}Z^\delta$ and $R_\alpha \to R_\alpha Z^\alpha$. Similar sequences are available on any homogeneous space, where an analogue of a local twistor bundle is an extension of two irreducible bundles in a BGG resolution [1] linked by a simple reflection.

The easiest Penrose transform on any complex homogeneous space is always that of the highest forms—the result is non zero in highest possible degree only, always the kernel of an invariant differential operator, resolved by further invariant operators and irreducible (over $sl(4,\mathbf{C})$ in the standard twistor case). Here of course we get self-dual Maxwell fields:

$$0 \to H^1(\Omega^3) \to \mathcal{O}_{A'B'}[-1] \overset{\nabla^{B'}_{A'}}{\longrightarrow} \mathcal{O}_{AA'}[-3] \overset{\nabla^{AA'}}{\longrightarrow} \mathcal{O}[-4] \to 0.$$

The naïve idea is to start from this and use the long exact sequences on cohomology coming from (1) to compute the cohomology of the rest. The first step is to compute the cohomology of the local twistor bundles and the second to use this in the long exact sequences.

Take the case of $\mathcal{T}^\alpha[-3]$. This is obtained by coupling the result for $\mathcal{O}[-3]$ to local twistor transport on Minkowski space, whilst the result for $\mathcal{O}[-3]$ is in turn obtained from $H^1(\Omega^3)$ by helicity lowering. Thus $H^1(\mathcal{T}^\alpha[-3])$ consists of solutions of the local twistor equation

$$0 \;=\; \nabla_A^{A'}\begin{pmatrix} \omega^B_{A'} \\ \pi_{A'B'} + \frac{1}{2}\epsilon A'B'\phi \end{pmatrix} \;=\; \begin{pmatrix} \nabla_A^{A'}\omega^B_{A'} - \delta^B_a\phi \\ \nabla_A^{A'}\pi_{A'B'} + \frac{1}{2}\nabla_{AB'}\phi \end{pmatrix}.$$

Here, $\omega_{AA'} \in \mathcal{O}_{AA'}$ is a one form, $\phi \in \mathcal{O}[-2]$ and $\pi_{A'B'} \in \mathcal{O}_{(A'B')}[-1]$ is a self dual two-form. The first of these equations simply fixes $\phi = \frac{1}{2}\nabla_A^{A'}\omega^A_{A'}$ and one is left with

$$\nabla^{A'}_{(A}\omega_{B)A'} = d^+\omega = 0 \quad \nabla_A^{A'}\pi_{A'B'} - \tfrac{1}{4}\nabla_{AB'}\nabla^c\omega_c = 0.$$

Since $d : \Omega^2_+ \to \Omega^3$ is onto $\ker d : \Omega^3 \to \Omega^4$, we can regard this last equation as fixing $\pi_{A'B'}$ (up to a self dual Maxwell field) and requiring $\Box\nabla^c\omega_c = 0$. Thus

$$H^1(\mathcal{T}^\alpha(-3)) = \text{ s.d. Maxwell } + \{\omega_c | d^+\omega = 0 = \Box\nabla^c\omega_c\}.$$

$H^1(\Omega^2)$ is then just the second term. This is a non trivial extension of two irreducibles, namely the anti-self-dual Maxwell fields (obtained via potentials mod gauge) and

$$\{f \in \mathcal{O} \mid \Box^2 f = 0)/\mathbb{C}$$

(obtained by letting $\omega_c = \nabla_c f$). Of course, this calculation checks with more standard methods. The cohomology of $\Omega^1, \mathcal{O}$ are equally easy to compute this way.

To carry this out for general complex homogeneous spaces we have to overcome two difficulties. The first is to obtain the cohomology of local twistor bundles. This is done by observing that helicity raising and lowering (or the *translation principle*) commutes with taking cohomology and that the result splits into a direct sum of (χ-primary) parts under the action of the center of $\mathcal{U}(\mathfrak{g})$—i.e. under Casimir operators. One has to perform two translations, by a finite dimensional representation and its dual. The first translates to *singular* character and the second out again to regular character. Each time, we also project out only one of the χ-primary parts. Let's label these two operations ψ_α, ϕ_α—α is a simple root for $\mathfrak{g}$ and the finite dimensional module is just $F(\lambda)$ where λ is the fundamental weight dual to α. Vogan has an algorithm for calculating the $\phi_\alpha\psi_\alpha$ on irreducibles, which succeeds by the Kazhdan–Lusztig conjectures. This reduces the problem to (not too difficult) combinatorics. The second difficulty is to understand the maps in the long sequence. Here, translation again comes to the rescue. It turns out that we can often pick α so that for a given homogeneous bundle $\mathcal{F}$, $\psi_\alpha\mathcal{F} = 0$. Then $\psi_\alpha H^\bullet(\mathcal{F}) = 0$ too. So any irreducible *not* annihilated by ψ_α can't occur $H^\bullet(\mathcal{F})$! This data is easy to deduce from Hasse diagrams (see my other article in this TN).

Conjecture: This is enough (with Schur's lema) to calculate the maps in the long exact sequence (and so to compute the Penrose transform of any homogeneous bundle).

I've checked this to be true in many cases, one involving so(12)!

A slightly stronger conjecture is that if an irreducible occurs in H^i and $\phi_\alpha\psi_\alpha H^i$ then it is mapped non trivially from one to the other. This is true in all the examples I know. If it is true in general it should follow that the Penrose transform will detect all non-trivial homomorphisms of Verma modules.

[1] Baston/Eastwood: *The Penrose transform—its interaction with representation theory*. O.U.P. (1989).

Rob Baston

Cohomological contours and cobord maps

Introduction

In [2], we asked how to tell whether a given contour κ is cohomological, that is, whether κ treats the twistor functions on the 'outside' part of the diagram properly as cohomology classes. If A and B are a pair of ears in a twistor diagram (i.e. a pair of planes on which a twistor function blows up) then by duality, κ is cohomological with respect to (A, B) iff $\kappa = \partial_* \lambda$ where ∂_* is the Mayer-Vietoris connecting homomorphism from the complement of $A \cup B$ to the complement of $A \cap B$. By the main Lemma of [2] (reproduced as Lemma 1 below), this somewhat impractical criterion is equivalent to the following

> CRITERION: κ is cohomological if *either* there is a λ with
>
> $$\delta_a(\lambda) = \kappa \text{ and } \delta_b(\lambda) = 0$$
>
> *or* there is a μ satisfying these conditions with a and b interchanged.

Here $\delta_{a,b}$ are the cobord maps corresponding to A, B. The reason this is useful is that the cobord maps are dual to the external S^1 integrals which are used all the time in twistor diagram theory.

APH has suggested that for any twistor diagram whose evaluation does not require the use of a contour with boundary, one has the much stronger $\kappa = \delta_a\delta_b$(something). Moreover, he has an example of a cohomological contour (for the scalar product diagram for spin 1) whose evaluation requires the use of a boundary contour (or some other such device) for which the *relative* version of the above criterion holds, but for which the condition of this paragraph *fails*.

In this article we show that the criterion of [2] does indeed carry over to the relative situation and we also prove APH's conjecture about contours without boundary by relating it to the main result of [1]. We make significant use of the lemma mentioned above, which we restate here in a generalized form:

Lemma 1 *Let X be a complex manifold and let S_1, S_2 be complex submanifolds of (complex) codimensions p_1, p_2, in general position so that $S = S_1 \cap S_2$ is a complex submanifold of complex codimension $p = p_1 + p_2$.*

Consider the two Leray sequences

$$\begin{array}{ccccccc}
\to & H_{i+1}(X - S) & \xrightarrow{\cap_a} & H_{i-2p_1+1}(S_1 - S_2) & \xrightarrow{\delta_a} & H_i(X - S_1) & \to \\
 & \downarrow & & \| & & \downarrow & \\
\to & H_{i+1}(X - S_2) & \xrightarrow{\cap_b} & H_{i-2p_1+1}(S_1 - S_2) & \xrightarrow{\delta_b} & H_i(X - S_1 - S_2) & \to
\end{array}$$

Then the composite $\delta_b \cap_a$ is equal to the Mayer-Vietoris connecting homomorphism

$$\partial_* : H_{i+1}(X - S) \longrightarrow H_i(X - S_1 - S_2).$$

Proof. This is exactly as in [2]: the essential point is that even though we have allowed the codimensions to exceed 1, the formula for the cobord map in terms of compactly supported differential forms is unchanged. □

Residues, cobord maps and the 'general procedure'

In [1] it was shown that if κ is a cohomological contour without boundary which evaluates a projective twistor diagram then κ is necessarily in the image of the cobord map

$$\Delta : H_v(\Lambda - \Sigma) \longrightarrow H_{6v-1}(\Pi - \Lambda - \Sigma) \qquad (1)$$

where v is the number of fields, Π is the product of v (projective) twistor/dual twistor spaces, Λ is the product of lines on which the fields are based and Σ is the singularity set of the kernel (interior) of the diagram. An element κ in the RHS is used to evaluate the diagram by *dotting* all the external fields together, multiplying by the kernel of the diagram and integrating over κ.

On the other hand, those who evaluate diagrams on a professional basis are accustomed to the use of a large number of 'small' (i.e. lower codimensional) cobord maps. We remarked in [1] that the methods had to be essentially equivalent and dual to the relationship between cupped and dotted forms. Using Lemma 1, we can now make this precise.

Proposition 1 *Let X, S_1, S_2 etc. be as in Lemma 1. Then the following diagram is commutative:*

$$\begin{array}{ccccc} H_*(S) & \xrightarrow{\delta_2} & H_{*+2p_2-1}(S_1 - S_2) & \xrightarrow{\delta_1} & H_{*+2p-2}(X - S_1 - S_2) \\ \| & & & & \| \\ H_*(S) & \xrightarrow{\Delta} & H_*(X - S) & \xrightarrow{\partial_*} & H_{*+2p-2}(X - S_1 - S_2) \end{array}$$

Here $\Delta, \delta_1, \delta_2$ are the obvious cobord maps and ∂_* is the Mayer-Vietoris connecting homomorphism.

Proof. By Lemma 1, ∂_* can be replaced by the composite

$$H_{*+2p-1}(X - S) \xrightarrow{\cap_a} H_{*+2p-1-2p_1}(S_1 - S_2) \xrightarrow{\delta_b} H_{*+2p-2}(X - S_1 - S_2).$$

Since $\delta_b = \delta_1$ (and $p - p_1 = p_2$) it is enough to show that the following diagram is commutative:

$$\begin{array}{ccc} H_*(S) & \xrightarrow{\delta_2} & H_{*+2p_2-1}(S_1 - S_2) \\ \downarrow \Delta & & \| \\ H_{*+2p-1}(X - S) & \xrightarrow{\cap_a} & H_{*+2p_2-1}(S_1 - S_2) \end{array}$$

But that is obvious. □

Theorem 1 *Suppose that the above twistor diagram with v (external) vertices has an evaluation by means of a contour without boundary. Then any evaluation coming from the 'general procedure' (1) can be implemented by integration over $2v$ circles (one for each ear) followed by integration over a contour in $\Lambda - \Sigma$.*

Proof. (i) By applying the Proposition to the case $X = CP^3$, each S_i equal to a plane, we conclude that the cobord associated to a line in CP^3 is the same as the composition of two cobords, one for each plane, and a Mayer-Vietoris map.

(ii) Similarly, by applying the Proposition with X equal to the product of v twistor spaces and each S_i equal to the product of $v-1$ twistor spaces and one projective line, we see that the 'fat' cobord map Δ of (1) coincides with a cobord for each line and $v-1$ Mayer-Vietoris maps.

Combining (i) and (ii), we complete the proof.□

This theorem brutally exhibits the strengths and weaknesses of the 'general procedure'. It shows that *any cohomological evaluation by integration over a closed contour factors through taking residues at the external lines.* That is the strength of the procedure: its weakness is that this residue is very often zero for interesting twistor diagrams; for such diagrams, boundary contours or some non-trivial extension of the class of functionals considered in [1] are plainly needed.

A criterion for cohomological boundary contours

When is a boundary contour cohomological? As in the non-boundary case, a contour is cohomological if it lies in the image of a certain Mayer-Vietoris map. As we remarked above, this is not much use. However, with the aid of the following relative version of our basic lemma, we can get a practical criterion for a boundary contour to be cohomological.

Lemma 2 *Let X, S_1, S_2 etc. be as in Lemma 1. In addition, let F be a closed submanifold in general position. Consider the two Leray sequences*

$$\begin{array}{ccccccc}
\to & H_{i+1}(X-S,F) & \xrightarrow{\cap_a} & H_{i-2p_1+1}(S_1-S_2,F) & \xrightarrow{\delta_a} & H_i(X-S_1,F) & \to \\
 & \downarrow & & \| & & \downarrow & \\
\to & H_{i+1}(X-S_2,F) & \xrightarrow{\cap_b} & H_{i-2p_1+1}(S_1-S_2,F) & \xrightarrow{\delta_b} & H_i(X-S_1-S_2,F) & \to
\end{array}$$

Then the composite $\delta_b \cap_a$ is equal to the Mayer-Vietoris connecting homomorphism

$$\partial_* : H_{i+1}(X-S,F) \longrightarrow H_i(X-S_1-S_2,F). \qquad (2)$$

Proof. Recall that an element of $H_i(X, F)$ is represented by a pair of compactly supported forms (α, α') such that

$$\alpha \in \Lambda_c^{\dim X - i}(X),\ \alpha' \in \Lambda_c^{\dim X - i - 1}(F),\ d\alpha = 0,\ \alpha|F = d\alpha'. \tag{3}$$

(Here and throughout, '$|F$' means 'pull-back to F'.)

Let D be a tubular neighbourhood of $S_1 - S_2$, relatively compact in $X - S_2$; let j be the inclusion of $S_1 - S$ in $X - S$; let $\pi : D \to S_1$ be the projection which we may suppose carries points of $F \cap D$ to points of $F \cap (S_1 - S_2)$. Then there exists a homotopy operator H such that

$$\pi^* j^* u - u = (dH + Hd)u \text{ and } (Hu)|F = H(u|F) \tag{4}$$

for all forms u on D.

Let β be a positive smooth bump function on $X - S$, equal to 0 near $S_1 - S$ and equal to 1 in a neighbourhood of $X - D$. Then we have the following formulae:

$$\begin{aligned}
\partial_*(\alpha, \alpha') &= (\alpha \wedge d\beta, \alpha' \wedge (d\beta|F)); \\
\cap_a(\alpha, \alpha') &= (j^*(\alpha), j^*(\alpha')); \\
\delta_b(\gamma, \gamma') &= (\pi^*(\gamma) \wedge d\beta, \pi^*(\gamma') \wedge (d\beta|F)).
\end{aligned}$$

So

$$\begin{aligned}
\delta_b \cap_a (\alpha, \alpha') &= (\pi^* j^*(\alpha) \wedge d\beta, \pi^* j^*(\alpha') \wedge (d\beta|F) \\
&= (\alpha \wedge d\beta + dH\alpha \wedge d\beta, \alpha' \wedge (d\beta|F) + Hd\alpha' \wedge (d\beta|F) + dH\alpha' \wedge (d\beta|F)) \\
&= \partial_*(\alpha, \alpha') + (dH\alpha, (H\alpha|F) + dH\alpha') \wedge d\beta
\end{aligned}$$

using both parts of (4) and (3). To complete the proof, it suffices to note that the 'error term' in the equation above represents zero in relative homology by definition. □

Thus the criterion of TN28, restated in the Introduction, holds also for boundary contours: thus some of APH's standard procedures (reduction to CP^2 integrals) are justified.

What is still missing is a relative version of the 'general procedure' of [1]. Any offers, anyone?

Stephen Huggett M. A. Singer

References

[1] Huggett & Singer, Projective twistor diagrams and relative cohomology, Trans. A.M.S. (to appear, at last, in November 1990).

[2] Huggett & Singer, Cohomological residues, TN28.

Some new boundary-contour integrals

These notes fill in some gaps in the theory of contours for twistor diagrams. First, there's a spinor integral with two boundaries which doesn't seem to have been studied before, namely:

$$\frac{1}{2\pi i}\oint_{\xi.\beta=0,\ \eta.\delta=0}\frac{D\xi\wedge D\eta}{(\xi.\eta)(\eta.\gamma)(\xi.\alpha)} \qquad (1)$$

To construct a contour, first take the special case $\alpha = \gamma$, and coordinates

$$\xi^A = x\alpha^A + \beta^A, \qquad \eta_A = y\alpha_A + \beta_A$$

so the integral becomes

$$\frac{1}{2\pi i}(\alpha.\beta)^{-1}\oint_{x=0,\ y\alpha.\delta+\beta.\delta=0} dx\, dy\, (x-y)^{-1}$$

In this form it's immediate that there is a contour with the topology of an annulus yielding $\dfrac{\beta.\delta}{\alpha.\beta\ \alpha.\delta}$

By a power series expansion, or otherwise, the general case can be evaluated as

$$(\alpha.\gamma)^{-1}\log\left(\frac{\alpha.\beta\ \gamma.\delta}{\alpha.\delta\ \gamma.\beta}\right) = \int_0^1 \frac{\beta.\delta\ du}{\alpha.\gamma\,\beta.\delta\, u - \alpha.\delta\,\beta.\gamma}$$

The contour can be thought of as a sphere with two holes in it. One hole has boundary on $\xi.\beta=0$, and allows the singularity in $\eta.\gamma$ to poke through it; similarly for the other. If one of these singularities is absent, the corresponding hole can be filled in with a 'cap' (which makes no difference to the answer.) Thus neither pole is essential to the contour. This can be seen explicitly by evaluating a different form over the same contour, viz.

$$\frac{1}{2\pi i}\oint_{\substack{\xi.\beta=0\\ \eta.\delta=0}}\frac{D\xi\wedge D\eta}{(\xi.\eta)^2}\left(\frac{\eta.\nu}{\eta.\gamma}\right)\left(\frac{\xi.\kappa}{\xi.\alpha}\right) = \frac{\nu.\beta}{\gamma.\beta}\frac{\delta.\kappa}{\delta.\alpha} + \frac{\alpha.\kappa\ \nu.\gamma}{\alpha.\gamma}\left(\frac{\beta.\delta}{\delta.\alpha\ \beta.\gamma} + \frac{1}{\alpha.\gamma}\log\left(\frac{\alpha.\beta\ \gamma.\delta}{\alpha.\delta\ \gamma.\beta}\right)\right)$$

Then in the cases where $\nu=\gamma$ or $\kappa=\alpha$ we can 'cap' one hole and recover spinor integrals already well-known*

$$\frac{1}{2\pi i}\oint_{\eta.\delta=0}\frac{D\xi\wedge D\eta}{(\xi.\eta)^2}\frac{\xi.\kappa}{\xi.\alpha} = \frac{\delta.\kappa}{\delta.\alpha}\ ; \qquad \frac{1}{2\pi i}\oint_{\xi.\beta=0}\frac{D\xi\wedge D\eta}{(\xi.\eta)^2}\frac{\eta.\nu}{\eta.\gamma} = \frac{\nu.\beta}{\gamma.\beta}$$

If both $\nu=\gamma$ and $\kappa=\alpha$ then both caps can be put on and we recover $\displaystyle\frac{1}{2\pi i}\oint\frac{D\xi\wedge D\eta}{(\xi.\eta)^2} = 1$

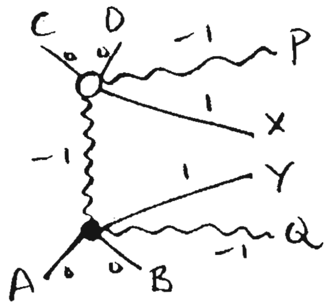
There is also an analogous contour for

$$\oint_{\substack{\zeta.\beta=0,\ \eta.\delta=0 \\ \zeta.\eta=0}} \frac{D\zeta\wedge D\eta}{(\eta.\gamma)^2(\zeta.\alpha)^2} \qquad (2)$$

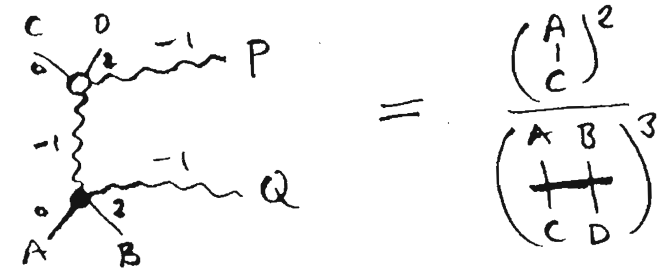
To see this, use the same special case and the same coordinates, then the contour is readily specified as the interior of a triangle bounded by the three given surfaces. The result in the general case is

$$\int_0^1 \frac{(\beta.\delta)^2\, u\, du}{(\alpha.\gamma\,\beta.\delta\, u - \alpha.\delta\,\beta.\gamma)^2}$$

These contours for spinor integrals imply (via cobordism) contours for certain twistor integrals. In particular, (2) induces a contour for:



with the property that (AB), (CD) are treated cohomologically, whilst the X and Y poles are not essential.
Because these poles can be 'filled in', we can operate with $(X.\partial_D)^2(Y.\partial_B)^2$
and still have a non-zero integral: in fact explicit calculation yields



But the RHS is just the inner product for the two spin-1 elementary states. This means that the spin-1 inner product *can* be represented as a twistor integral which reduces to a spinor integral, i.e. with a contour that fibres as

$(S^1)^4 \times$ 

but to do this we have to introduce these extra boundaries.

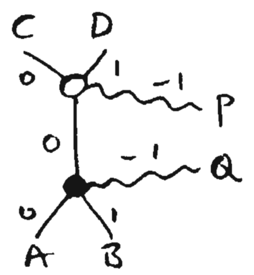
Since the position of the boundaries makes no difference to the answer, it looks as though they can be removed by adding some 'caps'; and indeed they can within the *entire* twistor integral, but at the cost of abandoning the fibration which reduces it to a spinor integral. In fact the contour (without any extra boundaries) can be specified as (solid ball in R^4 x $(S^1)^2$). This integral is of course the simplest example of a cohomological contour which cannot be realised as (a spinor integral x $(S^1)^4$) (see the article by SAH and MAS in this **TN**).

These new spinor integrals are also useful by guiding the analogous construction in CP^2. That is, we study

$$\oint_{\substack{b.x=0\\ d.y=0}} \frac{D^2x\wedge D^2y}{(x.y)(c.y)^2(a.x)^2}\;;\quad \oint_{\substack{b.x=0\\ d.y=0\\ x.y=0}} \frac{D^2x\wedge D^2y}{(c.y)^3(a.x)^3}$$

where x^a, y_a are elements of CP^2, CP^{2*}.

The construction of contours is not quite as immediate as in the spinor case but can be done. We can then use them to induce contours for the twistor diagrams



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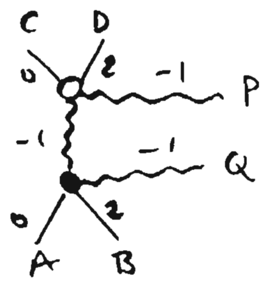


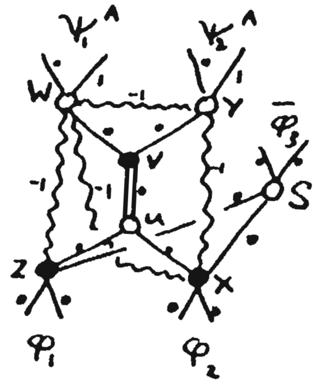
which are *not* cohomological in (AB), (CD). The first of these contours was shown to exist by SAH (D. Phil. thesis, 1980) but until now we haven't had a direct construction for it. The second is closely related. Explicit evaluation of the integrals shows agreement with the results previously derived by limiting techniques*). These contours are important in the theory of inhomogeneous diagrams*. There are further generalisations and deductions from these new constructions which, put together with SAH and MAS's recent work, should cap most of the holes in the theory of elementary twistor diagrams.

Andrew Hodges

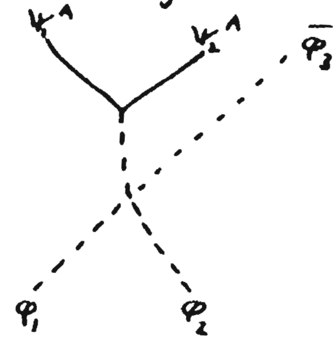
*APH, Proc. R. Soc. Lond. A 397, 341-374 (1985)

Some Yukawa Interaction Diagrams

In $\mathbb{T}\mathbb{N}$ 29 A.P.H was led to consider the twistor diagram

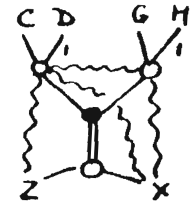


as the translation of the Feynman diagram $\longleftrightarrow$



To integrate the top part of this diagram explicitly one uses a boundary contour for both Y and W separately (see R.P. in A.T.T.) After an integration by parts, the $\check{u}$-pole can be surrounded by an S^2-contour giving, for elementary states:

$$F(\acute{z},\acute{x}) = $$

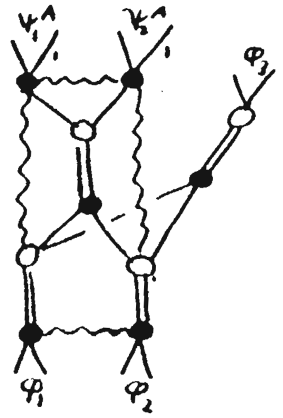


$$= \int_0^\infty \frac{\overline{ZX}\,\rho\,(\overline{CGGHZX} + \overline{GCZX}\,\overline{GH}\,\rho)\,d\rho}{\overline{GHZX}\,(\overline{CD}\,\overline{GHZX} + \overline{CD}\,\underset{Z\quad X}{\sqcup\!\sqcup}\,\overline{GH}\,\rho + \overline{CDZX}\,\overline{GH}\,\rho^2)^2}$$

$F(\acute{z},\acute{x})$ is a function of the field points $\overleftrightarrow{GH}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{ZX}$ only. It is exactly $\Box^{-1}(\psi_{1A}(x)\,\psi_2^{\ A}(x))$ in space-time.

Now one completes the S-integral, leaving a spinor integral in $\overleftrightarrow{ZX}$.

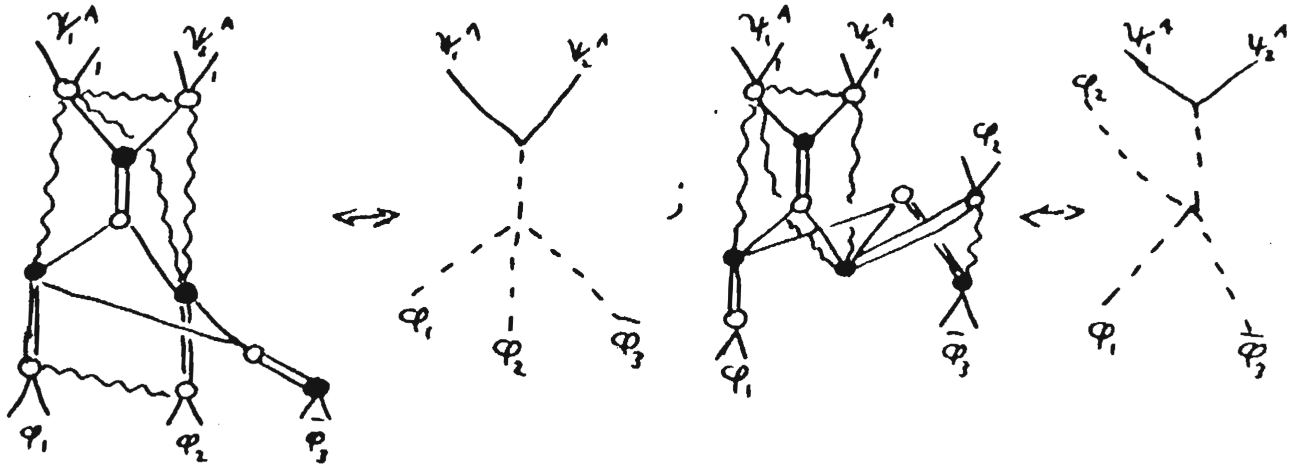
One can recast this into the form



which fits in better with the skeleton diagram structure of $\mathbb{T}\mathbb{N}$ 29.

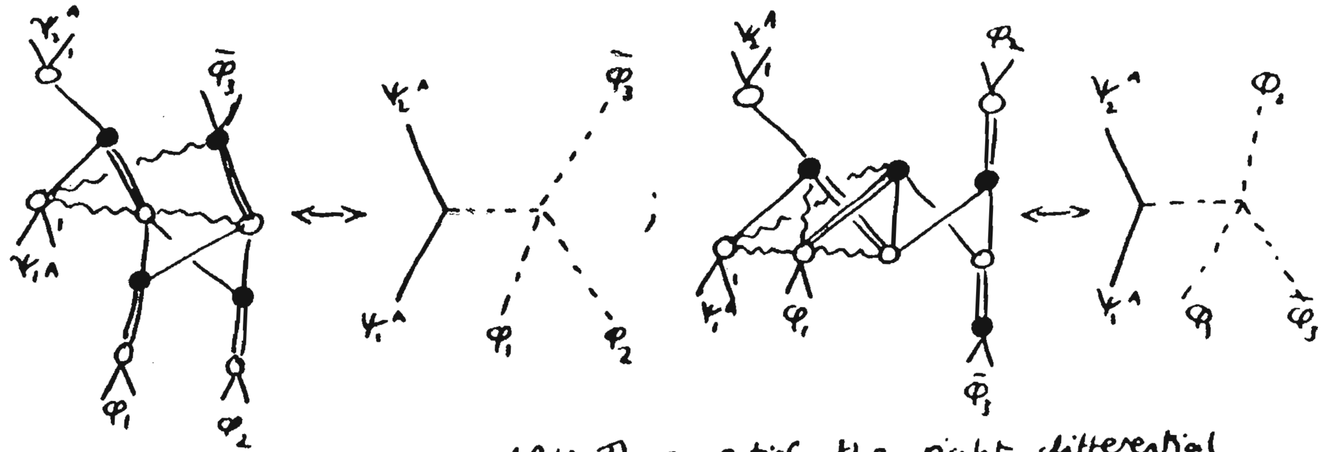
The other channels for the time-like propagator can now be written down with reference to the channels of the single box. Here one has to take seriously the identification of φ and $\bar{\varphi}$ with dual twistor functions, and twistor functions.

Thus one finds



These three channels have all been calculated explicitly, and agree with the Feynman diagrams.

The channels with a space-like propagator of this Feynman diagram are harder to check explicitly. The main difficulty being that the space-like propagator, with the two spinors, has less coincidences available, and thus is harder to integrate. Nevertheless the correspondences



have been written down by A.P.H. These satisfy the right differential properties, and are interpretable as scattering amplitudes. Work is proceeding on verifying these correspondences through calculation.

All these twistor diagram representations of the Feynman diagrams fit into the skeleton diagram picture (A.P.H in TMP). Thus the twistor diagrams are crossing symmetric in the same sense as the Feynman diagrams. Only the choice of boundary lines - and hence the region of integration - differs between the channels.

L.O'D.

A.P.H.

Extended Regge Trajectories: 12 years after

We wish to give here an extremely brief summary of the status of extended Regge trajectories[1]. We shall present a more detailed report after the new 1990 Particle Data Tables are out.

There are 10 resonances (5 mesons, 5 baryons) whose spin-parity (J^P) were unknown and predicted in our paper in 1978. Of these ten, five (four with confidence) were assigned negative j values. Nine of these J^P assignments are confirmed by the 1988 Tables,[2] and the remaining one is wrong. In the table below we set out the comparison:

1976	Predicted $J^P(j)$	1988
$D(1285)$ 0^+ $(A)^+$	1^+ (1)	$f_1(1285)$ $0^+(1^{++})$
$E(1420)$ 0^+ $(A)^+$	1^+ (-2)	$f_1(1420)$ $0^+(1^{++})$
$A_4(1900)$ 1^- $(?)$	2^- (-3)	$\pi_2(2100)$ $1^-(2^{-+})$
$\rho(2100)$?	3^+ (-4)	$\rho_2(2250)$ $1^+(3^{--})$
$K_N(1700)$ $\frac{1}{2}$ $(?)$	1^- (-2)	$K^*(1715)$ $\frac{1}{2}$ (1^-)
$N(2650)$ $\frac{1}{2}$ $(?^-)$	$\frac{13}{2}^+$ $(\frac{13}{2})$	$N(2700)$ $\frac{1}{2}$ $(\frac{13}{2}^+)$
$\Delta(2160)$ $\frac{3}{2}$ $(?)$	$\frac{7}{2}^-$ $(\frac{7}{2})$	$\Delta(2200)$ $\frac{3}{2}$ $(\frac{7}{2}^-)$
$\Delta(2850)$ $\frac{3}{2}$ $(?^+)$	$\frac{15}{2}^+$ $(-\frac{17}{2})$	$\Delta(2950)$ $\frac{3}{2}$ $(\frac{15}{2}^+)$
$\Lambda(2350)$ 0 $(?)$	$\frac{9}{2}^+$ $(\frac{9}{2})$	$\Lambda(2350)$ 0 $(\frac{9}{2}^+)$
$\Xi(1820)$ $\frac{1}{2}$ $(?)$	$\frac{3}{2}^-$ $(\frac{3}{2})$	$\Xi(1820)$ $\frac{1}{2}$ $(\frac{3}{2}^-)$

We note that not only have data changed considerably during the 12 years, but all mesons have been renamed!

Roger Pearson
Tsou Sheung Tsun

References

1. Roger Pearson, George Sparling and Tsou Sheung Tsun, J. Phys A 11 (1978) L231–L235.
2. Particle Properties Data Booklet, April 1988.

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