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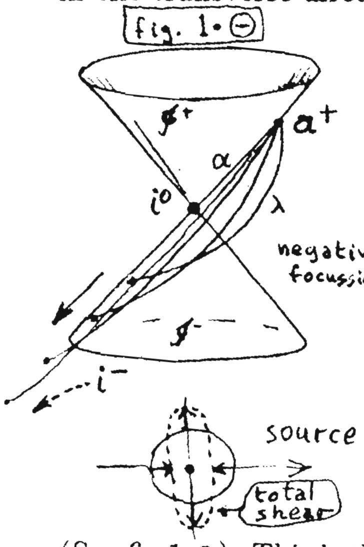
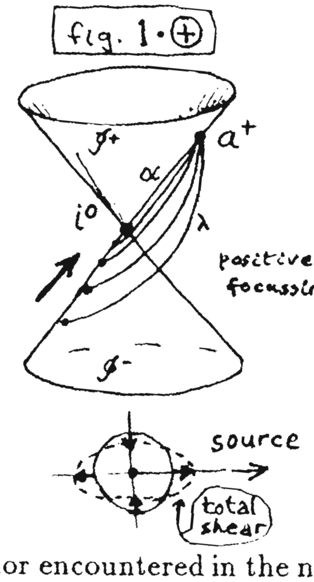
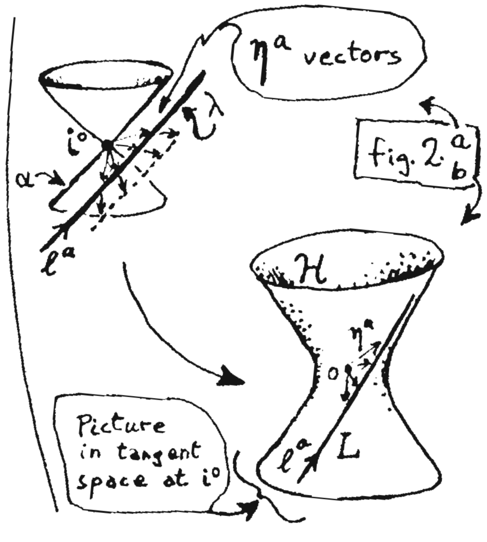
Mass Positivity from Focussing and the Structure of i^o

Recently, one of us (RP in **TN** 30, [1]) outlined an argument to show that in asymptotically simple space-times, satisfying a certain assumption, the mass is non-negative. The assumption –the *null conjugate point condition*– requires that every endless null geodesic contain a pair of conjugate points. This condition is physically reasonable. It is implied by completeness of null geodesics, a weak energy condition ($T_{ab}l^a l^b \geq 0$ if $l_a l^a = 0$) and genericity. The purpose of this contribution is to examine more closely what it is about the structure of spatial infinity, i^o , that the argument establishes. In particular, we will be able to establish that the mass in question is indeed the *ADM mass* at i^o . More precisely, our main result is that the ADM 4-momentum P_a of an asymptotically simple space-time satisfying the null conjugate point condition is necessarily a future-causal (or zero) 4-vector at i^o . (In this article, the assumption of asymptotic simplicity will include, in addition to asymptotic flatness at null infinity, that at spatial infinity. Thus, we assume that the space-time is AEFANSI in the sense of [2] and that the parity condition of [3] is satisfied at i^o .)

Let us begin by reviewing the result presented in [1]. Let $\mathcal{M}$ be an asymptotically simple space-time. Fix a point a^+ on the future null infinity $\mathscr{I}^+$ of $\mathcal{M}$. Let it lie on the generator α of $\mathscr{I}$. The past-directed light rays from a^+ will be said to *focus negatively* if they meet $\mathscr{I}^-$ in a family of points which recede indefinitely into the past (i.e., towards i^-) as these rays approach α. Let us suppose that this occurs. Then, if we examine the light rays neighbouring a given ray λ through a^+, as λ approaches α, the total shear of these rays along λ has the form of a convergence in the radial direction and a divergence in the transverse direction.







(See fig 1.⊖). This is the behavior encountered in the negative mass Schwarzschild solution. In the positive mass Schwarzschild space-time, the situation is just the opposite. Now, we encounter *positive focussing.* In this case, light rays from a^+ meet $\mathscr{I}^-$ in a family of points that approaches i^o as the rays approach α, there is a divergence of rays in the radia direction and convergence in the transverse direction. Thus, if we move in radially, the ray spray away from each other as i^o is approached, whereas transversally they pinch toward one another. (See fig 1.⊕.) Now, as the null geodesic λ gets closer to α, various non-linea terms die off and the shear is given by the integral, $\int_\lambda ds\, C_{abcd}l^a l^c \eta^b \eta^d$, evaluated along λ, where l^a is a parallelly propagated tangent to λ, s the corresponding affine parameter, and where η^a is the *radial* connecting vector. Thus, if negative focussing occurs, the integral would be positive and if positive focussing occurs, the integral would be negative. It was shown in [1] that, *if $\mathcal{M}$ satisfies the null conjugate point condition, negative focussing cannot occur.* Hence, in this case, the integral is necessarily non-positive for all geodesics λ originating at some point a^+ on $\mathcal{I}^+$ which are sufficiently close to the generator α of $\mathcal{I}$ on which a^+ lies. In this article, we show that this result in turn implies that the ADM 4-momentum is a (future pointing) causal vector. Thus, the intuition derived from the Schwarzschild solution is indeed valid more generally.

Let us recall the structure available at i^o . The asymptotic conditions of [2] imply that the tangent space at i^o is well-defined and carries a (universal) Minkowskian metric of signature $(-+++)$. Further, along any C^1 curve with tangent vector η^a at i^o , $\Omega^{\frac{1}{2}} C_{abcd}$ admits a limit at i^o , where C_{abcd} is the Weyl tensor of the unphysical metric g_{ab}. We can decompose the limit into its electric and magnetic parts using the unit space-like vectors η^a at i^o , and thus acquire two *symmetric, traceless* tensor fields $E_{ab}(\eta)$ and $B_{ab}(\eta)$ on the hyperboloid $\mathcal{H}$ of unit space-like vectors at i^o . (Note that, by their definition, the two fields are tangential to $\mathcal{H}$). Let us focus on the asymptotic electric field, E_{ab}. It satisfies the field equation $D_{[a}E_{b]c} = 0$ on $\mathcal{H}$, where D is the derivative operator compatible with the natural metric $h_{ab} = g^o{}_{ab} - \eta_a \eta_b$ on $\mathcal{H}$, where $g^o{}_{ab}$ is the Minkowski metric in the tangent space at i^o . The information about the ADM 4-momentum P_a is contained entirely in E_{ab}:

$$P_a V^a := -\tfrac{1}{8\pi} \oint_S dS^a\, E_{ab} V^b, \tag{1}$$

where V^a is any vector in the tangent space at i^o and S is any 2-sphere cross-section of $\mathcal{H}$. (The field equation and the trace-free property of E_{ab} imply that it is divergence-free while the projection of V^a into the hyperboloid (forced by its contraction with E_{ab} in the integrand) is a conformal Killing field on $\mathcal{H}$, whence the surface independence of (1).)

The field equation on E_{ab} also implies that it admits a scalar potential, E:

$$E_{ab} = D_a D_b E + E h_{ab} \tag{2.a}$$

In any given conformal completion, the potential E can in fact be constructed explicitly from the asymptotic (unphysical) Ricci tensor. The fact that E_{ab} is trace-free implies that E must satisfy the (tachyonic) massive scalar field equation:

$$D^a D_a E + 3E = 0 \tag{2.b}$$

on $\mathcal{H}$. In the Schwarzschild space-time with 4-momentum $P_a = m t_a$, with $t.t = -1$, E is given by:

$$E = m\, \frac{1+2(t.\eta)^2}{\sqrt{1+(t.\eta)^2}} \tag{3}$$

More generally, in physically interesting situations, the asymptotic magnetic field B_{ab} vanishes on $\mathcal{H}$ and the leading terms in the *physical* metric are dictated entirely by E:

There exists a coordinate system in terms of which the physical metric has the asymptotic form:

$$d\hat{s}^2 = (1 + \tfrac{E}{\rho})^2 d\rho^2 + \rho^2 \left(h^o{}_{\mathbf{ab}} + \tfrac{h^1{}_{\mathbf{ab}}}{\rho} + \tfrac{h^2{}_{\mathbf{ab}}}{\rho^2} + \ldots\right) d\phi^{\mathbf{a}} d\phi^{\mathbf{b}}, \tag{4}$$

with $h^1{}_{\mathbf{ab}} = Eh^o{}_{\mathbf{ab}}$. Here $h^o{}_{\mathbf{ab}}$ is the metric on the unit time-like hyperboloid in Minkowski space. Thus, $(\rho, \phi^{\mathbf{a}})$ should be thought of as "asymptotically hyperboloidal" coordinates.

It is easy to check that the space of solutions to the equation $D_a D_b \bar{E} + \bar{E} h_{ab} = 0$ is precisely 4-dimensional. The solutions are of the form $\bar{E} = K_a \eta^a$, where K_a is a fixed vector in the tangent space of i^o . Thus, there is some gauge freedom in the choice of the potential; we can add to the natural potential E of E_{ab} any $\bar{E}$ without changing the value of the field E_{ab}. This freedom is intertwined with the fact that if one uses only the asymptotic conditions of [2], there is some ambiguity in the conformal completion at i^o . Given a completion, one can obtain a four parameter family of inequivalent ones by logarithmic translations. In the physical space language, there are the transformations of the type:

$$x^{\mathbf{a}} \to x^{\mathbf{a}} + K^{\mathbf{a}} log \rho, \tag{5}$$

where $x^{\mathbf{a}}$ are asymptotically Cartesian coordinates, $\rho^2 = x^{\mathbf{a}} x_{\mathbf{a}}$ and where $K_{\mathbf{a}}$ are constants. The new completions are C^1 related at i^o . Therefore, i^o and the tangent spaces in the two completions are naturally identified. Under this identification, the field E_{ab} of one completion is mapped to that of the second. The potentials E, however, are not preserved. On $\mathcal{H}$ we have:

$$E \to E + K_a \eta^a \qquad \text{and} \qquad E_{ab} \to E_{ab} \tag{6}$$

Since the field is unaffected, so is the ADM 4-momentum (and, to next order, also angular momentum). Thus, the logarithmic translations may be thought of as "gauge" in this framework. In a large class of space-times, this gauge freedom can be eliminated. Suppose, as in [3], that the electric field E_{ab} is reflection symmetric on $\mathcal{H}$. Then, we can demand that the potential should also be reflection symmetric. (Note that E of the Schwarzschild solution (eq 3) automatically satisfies the condition. In Minkowski space the requirement singles out the potential $E = 0$.) This requirement selects a unique potential and hence removes the logarithmic ambiguity in the completion. As a part of the boundary conditions at i^o we assume that E_{ab} is reflection symmetric and work in a completion in which the potential E is also even under reflection.

With this formalism at hand, let us return to the implication of [1] quoted at the end of the first para. As the null geodesic λ approaches the generator α in the completed space-time (fig 2.a), the connecting vectors η^a become, in the limit, the position vectors of points on a null geodesic (straight line) L in $\mathcal{H}$, and the tangent vector l^a is now parallelly propagated w.r.t. h_{ab} (fig 2.b). Since the argument of [1] tells us that integral of $C_{abcd}\eta^a\eta^c l^b l^d$ along λ is non-positive, in the limit we conclude that $\int_L ds\, E_{ab}l^a l^b \leq 0$. Using the expression (2) of E_{ab}, we have $E_{ab}l^a l^b = (l^a D_a)^2 E$, so that the last condition becomes:

$$\dot{E}^+ - \dot{E}^- \leq 0, \tag{7.a}$$

where $\dot{E} = l^a D_a E$ and $\pm$ denote, respectively, the values at the future and past (ideal) end points of L. Now, since E is reflection symmetric, the two terms on the left side of (6) add and we have:

$$\dot{E}^+ \leq 0. \tag{7.b}$$

To see the implication of this condition, let us examine the asymptotic form of E. Let us foliate the tangent space of i^o by a family of planes $t = const$ (with $t = -t^a\eta_a$, with t^a unit future-timelike) and consider the corresponding foliation of $\mathcal{H}$. Assuming that E admits a power series expansion of the type $\sum E^{(n)}(\theta,\phi)\, t^{-n}$, where n runs from some *finite* negative value to $+\infty$, the field equation implies that E must admit an asymptotic expansion of the following type:

$$E(t,\theta,\phi) = (a_0 + a_m Y_{1m})(\theta,\phi))t + \frac{a_m Y_{1m}(\theta,\phi)}{2t} + \frac{E^{(3)}(\theta,\phi)}{t^3} + ... \tag{8}$$

where $Y_{1m}(\theta,\phi)$ are the three $\ell = 1$ spherical harmonics. (We believe the required assumption is always satisfied in the reflection-symmetric case.) The condition $\dot{E}^+ \leq 0$ implies that the coefficient of the first term is non-negative and hence the 4-vector $\bar{P}_a = a_0 t^a + V^a$ at i^o –with t^a the above unit time-like vector orthogonal to the slices and the spatial vector V^a in the $t = 0$ slice, given by $V^a\eta_a = a_m Y_{1m}(\theta,\phi)$– is future-directed and causal. At first, we were misled into thinking that the coefficient of the first term is the mass-aspect at the future end of $\mathcal{H}$ and therefore the argument would show that the mass-aspect should be positive. This is incorrect. In fact, the electric field E_{ab} constructed from the leading order term (via eq (2.a)) vanishes identically whence the term makes no contribution whatsoever to the ADM 4-momentum integral. Rather, the mass-aspect is the third term, $E^{(3)}$ in the expansion. However, because we have restricted ourselves to even potentials E, it *does* follow that the ADM 4-momentum P_a constructed from the correct mass aspect is precisely given by the vector $\bar{P}_a$, which resides in the leading order term. Therefore, although we cannot conclude that the mass-aspect should be positive, (7.b) does indeed imply that the ADM-4-momentum is a causal, future-directed vector.

We conclude with two remarks. First, one can show that the vector space obtained by superposing the asymptotic mass-aspects of Schwarzschild solutions (3) (whose the 4-momentum is not restricted to be time-like) is dense in the space of all asymptotic mass-aspects $E^{(3)}(\theta,\phi)$ arising from smooth solutions to (2.b). We believe, furthermore, that the same is true of the entire solutions E everywhere on $\mathcal{H}$. Thus, in the reflection-symmetric case, one can work with superpositions of (3) without loss of generality. The second remark has to do with the leading term in the asymptotic expansion (8). The fact that this term diverges is an indication that there is a mis-match at i^o between the following two limits: sliding down a generator of $\mathcal{I}^+$, and, approaching i^o along a space-like direction and then making an infinite boost. The mis-match is a measure of the ADM 4-momentum. There is a similar mis-match in the way $\mathcal{I}^-$ is attached to i^o . This mis-match should show up in the metric coefficients rather than the curvature. If one allows E to acquire a non-symmetric part under reflection, one can, by a logarithmic translation, remove the mis-match between i^o and $\mathcal{I}^+$. However, then the mis-match with $\mathcal{I}^-$ is twice as big. There is something "cohomological" here: with one coordinate choice near i^o , it appears that $\mathcal{I}^+$ matches "smoothly" on to i^o , whereas for another coordinate choice near i^o it would be $\mathcal{I}^-$ that matches smoothly to i^o . The 4-momentum represents the mismatch between these two attempts at a smooth structure at i^o . It would seem that these two choices correspond to whether we use the intersections with $\mathcal{I}^+$ or with $\mathcal{I}^-$ to represent light rays in the space-time. There appears to be a relation to twistor theory here. A clearer treatment of this issue is needed to make further progress in a proper understanding of asymptotically flat space-times, e.g. along the lines initiated by Friedrich.

We have presented here only the overall picture. Some details are yet to be worked out fully. Also, the results can probably be generalized in a number of ways. A more complete account will appear elsewhere.

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Conformally Invariant Operators: Singular cases.

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Abstract

All invariant linear differential operators between bundles of singular weight on flat conformal manifolds are determined and shown to have analogues on general conformal manifolds, obtained by adding suitable curvature correction terms.