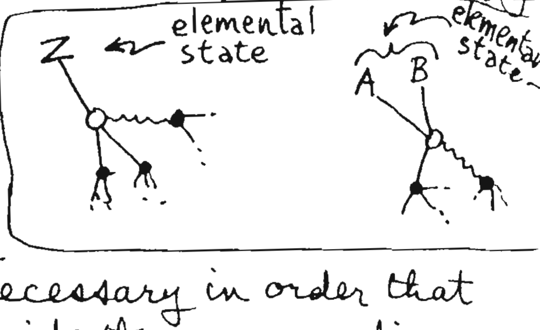
Twistor Theory for Vacuum Space-Times: a New Approach

A fundamental problem that has confronted the twistor programme since its earliest days has been to find the appropriate twistor theory for curved space-times, and most particularly *vacuum* space-times. Many of the stumbling blocks to different parts of the general programme seem to be rooted in the absence of an adequate union between the ideas of general relativity and twistor theory. (See R.P. in Bailey & Baston T.I.M.P. for a general survey of these problems and attempts at solution.) For some while it has seemed to me that one needs a concept of twistor with an intensely *non-local space-time description*. I shall present here a new concept of twistor that is, indeed, *very* non-local.

A difficulty with the asymptotic twistor / googly approach (with twistors defined at $\mathscr{I}$ only) has always been that twistors seem to have very little contact with the *finite* regions of space-time and with actual space-time points where, after all, the imposition of Einstein's vacuum equations $\mathbf{R}_{ab}=0$ is playing its role. The present suggestion may be regarded as a way of providing such contact.

The basic new ingredient is A.P.H.'s concept of *elemental state* (A.P.H. in $\mathbb{T}$ N 22 and p. 308–10 of F.A.T.T., Mason & Hughston) that, roughly speaking, is the "one-eared" version of the normal "two-eared" elementary state of standard twistor diagram theory. The two ears of the elementary states represent the two singularities which give the twistor function its H^1 character. That is necessary in order that the $\oint$ can be performed in order to provide the corresponding space-time field $\varphi_{\dots}$. This field is an H^0, i.e. it has a *local* space-time interpretation. The appropriate "$\oint$" on a "twistor function" with a single singularity (*one* ear) provides a *cohomological* space-time field. This is how the elemental states are described in space-time terms. There will be some covering (Stein) $\{\mathcal{U}_i\}$ of the relevant space-time region (say the complement of the α-plane represented by Z, in $\mathbb{C}\mathbb{M}^\#$), and a field $\varphi_{\dots}$ in each $\mathcal{U}_i$, providing a representative function of the cohomological object in question.



The idea here is to try to represent a twistor Z^α in terms of the (dual) "twistor function"

$$f(W_\alpha) = (Z^\alpha W_\alpha)_{-1} \qquad \left(\text{i.e. } Z \overset{-2}{\sim\!\sim\!\sim} W\right)$$

or perhaps, making use of A.P.H.'s insights with regard to twistor diagrams, something like

$$f(W_\alpha) = (Z^\alpha W_\alpha + k)_{-1} .$$

Here $(x)_{-1}$ is the <u>bracket factor</u> satisfying

$$\frac{d^2}{dx^2}(x)_{-1} = \frac{1}{x}$$

so we can take $(x)_{-1} = x\log x - x$ (or $x(\log x + \gamma - 1)$, where γ is Euler's constant — which may be more in line with A.P.H.'s prescriptions), or else $(x)_{-1} = x$ together with some rule that the contour has boundary on $x = 0$ (or $x + k = 0$), or else use a limiting argument with complex powers. If we try to integrate out the W_α-dependence to get a space-time "field", according to

$$\varphi_{\dots\dots} = \oint (Z^\alpha W_\alpha (+k))_{-1}\, d^2W$$

we end up with a <u>cohomological</u> field $\varphi_{\dots\dots}$.

If f had been an ordinary (H^1-type) twistor function of homogeneity $n = 1$, then it would represent a helicity $s = \frac{3}{2}$ massless field ($n = 2s - 2$, for dual twistor functions), but since our f has the wrong cohomological structure for this, it represents a <u>cohomological</u> (or worse!) helicity $\frac{3}{2}$ field. The special feature of helicity $3/2$ for our purposes here is that the equations work (Buchdahl) also in Einstein vacuum space-times ($R_{ab} = 0$) — but only if we represent the field appropriately by a <u>potential</u>, which I shall take in the form

$$\boxed{\nabla^{B'}_{B}\,\gamma_{AB'C'} = 0, \qquad \gamma_{AB'C'} = \gamma_{AC'B'}}$$

where we take

$$\boxed{\gamma_{AB'C'} \text{ modulo } \nabla_{AB'}\nu_{C'} \text{ with } \nabla^{B'}_{A}\nu_{B'} = 0.}$$

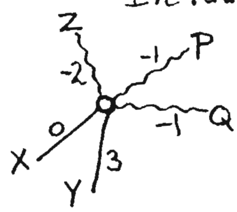
The consistency of these equations when $R_{ab} = 0$ is important to supergravity theorists, although the equations they use amount to the above without the symmetry condition on $\gamma_{\dots}$ and without the neutrino equation on $\nu_{\cdot}$. The two forms are locally equivalent (cf. Penrose & Rindler S & S-T, Vol. I, p.370). An immediate (Buchdahl) consistency condition on the field equation for $\gamma_{\dots}$ is $R_{ab} = \lambda g_{ab}$, while we require $R_{ab} = 0$ in order that $\nu_{C'}$ act as a gauge.

The use of these spin $3/2$ equations here is totally different from their use in supergravity theory, however. The proposal here is to represent each twistor Z^α by an appropriate cohomology class (or "worse"?) on a vacuum space-time $\mathcal{M}$, the representative functions being helicity $\frac{3}{2}$ fields as described above. Note that such a representation is <u>doubly</u> <u>non-local</u>, firstly because of the cohomology and secondly because the field must be represented as a potential modulo gauge. In flat space-time $\mathbb{M}$, we would have a local field quantity given by

$$\varphi_{A'B'C'} = \nabla^A_{A'}\gamma_{AB'C'}\,, \quad \text{satisfying } \nabla^{A'}_A\varphi_{A'B'C'} = 0\,,$$

but in curved space-time $\mathcal{M}$, neither does the field equation for $\varphi_{\ldots}$ follow nor is $\varphi_{\ldots}$ independent of γ.

In $\mathbb{M}$, we can find such representative $\varphi_{\ldots}$s and $\gamma_{\ldots}$s by appealing to a twistor diagram.



Here $X^\alpha = (ix^{AA'}\xi_{A'}, \xi_{A'})$ and $Y^\alpha = (ix^{AA'}\eta_{A'}, \eta_{A'})$ determine the field point (the line XY in $\mathbb{PT}$) and a local frame $\xi_{A'}, \eta_{A'}$ at that point; P^α and Q^α provide boundaries for the $\oint$ which determine the cohomological freedom. It appears to be the case that the field $\varphi_{\ldots}$ represents a <u>relative H^2</u>, which is taken relative to the α-plane determined by Z^α. The "cohomological status" of $\gamma_{\ldots}$ is more obscure, because of the presence of logarithmic branching in its domain (work in progress!). In $\mathcal{M}$, where α-planes do not generally exist, in place of an α-plane, we seem to require a "smeared out" or "thickened" α-plane.

To see how this works explicitly in $\mathbb{M}$, take P and Q at infinity, i.e. of the form $P^\alpha = (\rho^A, 0)$, $Q^\alpha = (\sigma^A, 0)$. Evaluating the above twistor diagram in $\mathbb{CM}$, we get a field (proportional to)

$$\varphi_{A'B'C'} = \frac{\pi_{A'}\pi_{B'}\pi_{C'}}{\sigma\cdot\omega(x)\;\rho\cdot\omega(x)} \qquad (\times\,\xi^{A'}\xi^{B'}\xi^{C'})$$

where ω^A and $\pi_{A'}$ are the spinor parts of Z^α at the origin O of $\mathbb{CM}$, with

$$\omega^A(x) = \omega^A - ix^{AA'}\pi_{A'}\,,$$

and $\sigma\cdot\omega(x) = \sigma_A\omega^A(x)$, etc. A potential for $\varphi_{A'B'C'}$ is given by

$$\gamma_{AB'C'} = \frac{\pi_{B'}\pi_{C'}\sigma_A\log(\rho\cdot\omega(x))}{\sigma\cdot\omega(x)}\,.$$

Of course, in $\mathcal{M}$, we cannot expect to find representatives of this form, and there is a problem as to how one should characterize those particular cohomological elements that represent a <u>twistor</u> (elemental state), rather than something more general (i.e.  rather than ). The best I can do so far is to go to $\mathbb{C}\mathscr{I}^+$ (or $\mathbb{C}\mathscr{I}^-$) and demand that it relate to an α-line in the same way as the above expressions in $\mathbb{CM}$. However it would be better if we can avoid direct reference to $\mathbb{C}\mathscr{I}$, if possible. (Work in progress.)

Note that if we had a "twistor" Z^α represented by $\gamma_{AB'C'}$ and a "dual twistor" W_α represented by $\zeta_{A'BC}$, then we might be able to define their scalar product by appropriately integrating (over some Pocchammer-type contour) the 3-form

$$\Delta = \zeta_{F'}{}^{ED}\gamma_E{}^{F'D'}e_{abcDD'}\,dx^a\wedge dx^b\wedge dx^c$$

(which is closed, and exact if either $\gamma_{\ldots}$ or $\zeta_{\ldots}$ is pure gauge. A.P.H. gives arguments to show that we would expect something like $\{\log(W\cdot Z/k) + \chi - 1\}W\cdot Z$ rather than just $W\cdot Z$