A measurement process in a stationary quantum system

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Conventionally, time appears in quantum mechanics as a c-number parameter on which physical quantities such as observables or states depend, but it is not itself an observable. This relic of classical physics must of course be regarded as a stopgap. Eventually quantum gravity will give us a unified theory of space-time and matter, from which in principle all dependence of observables on non-observables can be eliminated.

Some models that have this property have already been constructed, notably by DeWitt[1] as a way of making sense of canonical quantum gravity, and in a more hand-waving but more accessible way by Page and Wooters[2] considering simple systems with quantum clocks. All such models have in common a beautiful feature that is necessary but at first rather counter-intuitive: *The universe as a whole is at rest.* That is, the quantum state $|\Psi\rangle$ of the universe as a whole is an eigenstate of its Hamiltonian $\hat{\mathbf{H}}$. The reason why that is necessary is of course that otherwise the Schrödinger equation would give $|\Psi\rangle$, and therefore physical quantities, a dependence on an unphysical parameter t.

The observed phenomenon of quantities "changing with time" has nothing to do with any t-dependence. It is a correlation phenomenon. Although (or rather *because*) the universe is in an eigenstate of the Hamiltonian, it is not in an eigenstate of the position of hands on clocks, or of any other observable that the inhabitants might measure to tell them the time. Therefore it is in a superposition of such eigenstates, whose eigenvalues are the readings of clocks at different instants. Under the Everett interpretation this means that different instants co-exist. The division of the world into "instants of time" is just a special case of its division into Everett branches (often, somewhat misleadingly, called "parallel universes").

It seems to me that one of the most important types of time-dependence that we need to understand, both technically and physically, is that which occurs in measurement. Therefore I have tried to construct a model of a measurement process in a stationary universe. What follows is entirely heuristic. In other words, I am not trying to prove anything, only to answer the question, if time really is a quantum correlation phenomenon as just outlined, what might the state and Hamiltonian look like, and how would it all work out?

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[1] DeWitt, B.S. *Phys. Rev.* **160** 1967.

[2] Page, D.M. and Wooters, W. *Phys. Rev.* D27 1983.

As usual in the theory of measurement, let us divide the universe into three quantum subsystems:

(1) A system S_1, initially in state $|\psi\rangle$, in which the observable $\hat{X}$ with spectrum $\mathbf{Sp}(\hat{X})$ is to be measured. (What "initially" means will emerge below).

(2) An apparatus S_2, initially uncorrelated with S_1 and in a receptive state $|0\rangle$. S_2 has an observable $\hat{A}$, with $\mathbf{Sp}(\hat{X}) \subseteq \mathbf{Sp}(\hat{A})$, in which the measured value of $\hat{X}$ is to be stored.

(3) The rest of the universe, S_3.

Conventionally, the measurement would be described using a time-dependent Hamiltonian $\hat{H}(t)$ specifying an interaction between S_1 and S_2 with support only during the period $0 \le t \le T$ (say), and not involving S_3. If the measurement were perfect it would have the following effect during that period:

$$|x,\ 0\rangle \Rightarrow |x,\ x\rangle \qquad \left(\forall x \in \mathbf{Sp}(\hat{X})\right) \tag{1}$$

where the kets on the left and right of the "evolves-into" symbol "$\Rightarrow$" in (1) are the joint state of S_1 and S_2 immediately preceding and immediately following the measurement interaction, i.e. at times 0 and T respectively. The representation in (1) is in terms of simultaneous eigenstates of $\hat{X}$ and $\hat{A}$, labelled by the corresponding eigenvalues, and we are assuming for convenience that the receptive state of S_2 is the eigenstate $|0\rangle$ of $\hat{A}$.

If (as a further harmless idealization) $\hat{H}(t) = \hat{H}$, a constant operator, during the measurement,

$$|x,\ x\rangle = e^{i\hat{H}T}|x,\ 0\rangle \qquad \left(\forall x \in \mathbf{Sp}(\hat{X})\right). \tag{2}$$

That a Hamiltonian with this property exists follows from the unitarity of the required evolution (1).

The above description of a measurement process is incomplete in that it does not model time explicitly. The notions "before the measurement", "during the measurement" and "after the measurement", as well as "the duration of the measurement" are integral to the description and are all referred to as if they were observable quantities, but no quantum observable corresponding to any of these quantities is described — in fact there is no such observable in systems S_1 and S_2. Moreover system S_3 is described as not participating in the measurement process, but it is implicitly required that something outside S_1 and S_2 "switch the interaction on and off" at times 0 and T, to induce the necessary time-dependence in the dynamics of S_1 and S_2.

Now we follow Page and Wooters and extend the model to include time as an observable. Let $\hat{h}$ be the Hamiltonian of S_3, the "rest of the world", let $|0\rangle$ be some state of S_3 and define

$$|t\rangle \equiv e^{i\hat{h}t}|0\rangle \tag{3}$$

for all real t. Let $\mathcal{T}$ be a maximal set of real numbers such that the corresponding kets $|t\rangle$ are orthonormal for all $t \in \mathcal{T}$. If $|0\rangle$ and $\hat{h}$ have suitable properties, $\mathcal{T}$ will be a large set, approximating the real line in an appropriate physical sense, and there will exist an observable $\hat{T}$ of S_3 with $\mathbf{Sp}(\hat{T}) = \mathcal{T}$,

$$\hat{T} \equiv \sum_{t \in \mathcal{T}} t|t\rangle\langle t|, \tag{4}$$

which, as shall see, can serve as a time observable. That there can exist observables $\hat{T}$ and $\hat{h}$ with the properties just described may be shown by explicit construction: Given any set $\mathcal{T}$ of successive real numbers separated by intervals ε and any set $\{|t\rangle\}$ of orthonormal states of S_3 labelled by, among other things, the elements of $\mathcal{T}$, the observable

$$-\frac{i}{\varepsilon} log\left(\sum_{t \in \mathcal{T}} |t+\varepsilon\rangle\langle t|\right) \tag{5}$$

would serve as $\hat{h}$.

Suppose that S_1 starts in an arbitrary state $|\psi\rangle$ uncorrelated with S_2 or S_3.

$$|\psi\rangle = \sum_{x \in \mathbf{Sp}(\hat{X})} \lambda_x |x\rangle, \tag{6}$$

where

$$\sum_{x \in \mathbf{Sp}(\hat{X})} |\lambda_x|^2 = 1. \tag{7}$$

If the S_3-observable $\hat{T}$ as defined in (4) were really the "time" for systems S_1 and S_2, we should expect the universe as a whole, i.e. the system $S_1 \oplus S_2 \oplus S_3$, to be in a state something like

$$|\Psi\rangle = \sum_{x \in \mathbf{Sp}(\hat{X})} \lambda_x \sum_{t \in \mathbf{Sp}(\hat{T})} \mu_t \left[\theta(t<0)|x,\ 0\rangle + \theta(0 \le t \le T) e^{i\hat{H}t}|x,\ 0\rangle + \theta(t>T)|x,\ x\rangle\right] |t\rangle \tag{8}$$

where θ is the function that takes the value 1 when its argument is a true proposition, and 0 otherwise. $|\Psi\rangle$ is a fixed state with no time-dependence in the usual sense. Nevertheless if we choose to refer to the eigenvalues of $\hat{T}$ as "times", the (Everett) interpretation of $|\Psi\rangle$ that we read off from successive terms in its expansion (8) does describe motion:

> At times before 0, the apparatus observable $\hat{A}$ has the receptive value 0 and $\hat{X}$ is multi-valued. Between times 0 and T, $\hat{A}$ becomes multi-valued in a way that is correlated

with $\hat{X}$. After time T, in each branch $\hat{X}$ has its original value and $\hat{A}$ has that same value.

The complex amplitudes μ_t are arbitrary except that they satisfy

$$\sum_{t \in \mathbf{Sp}(\hat{T})} |\mu_t|^2 = 1 \tag{9}$$

in order to normalize $|\Psi\rangle$, and that presumably none of them vanishes. I have argued elsewhere that there is no physical reason why a state such as $|\Psi\rangle$ must be normalizable with respect to the sum over times t because the weight $|\mu_t|^2$ of an Everett branch corresponding to a particular time is not the probability of anything. However if we were to allow non-normalizable states we should have to go to the trouble of amending the Hilbert space formalism to give meaning to representations such as (8) when the sum in (9) is divergent. That is not worth doing for our present purposes because we shall not encounter any problem in normalizing $|\Psi\rangle$.

To say that $\mathbf{Sp}(\hat{T})$ should physically approximate the real line is to say that there should be many eigenvalues t of $\hat{T}$ in any interval over which quantities of interest vary significantly as functions of t. Therefore the sums over t in (8) and (9) should be replaceable by integrals.

We shall take

$$\mu_t = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha t^2} \tag{10}$$

where α is very small and positive so that μ_t varies very slowly over the interval of interest ($0 \le t \le T$) but nevertheless falls rapidly to zero as $t \to \pm\infty$. Any function with those properties would serve equally well in what follows.

Let $\hat{P}$ be the system-S_3 projection operator

$$\hat{P} = \sum_{\substack{t \in \mathbf{Sp}(\hat{T}) \\ 0 \le t \le T}} |t\rangle\langle t| \approx \int_0^T |t\rangle\langle t| \mathrm{d}t \tag{11}$$

for the time to lie between 0 and T, i.e. for the period of the measurement. Consider the Hamiltonian

$$\hat{\mathbf{H}} = \hat{H} \otimes \hat{P} + \hat{1} \otimes \hat{h} \tag{12}$$

for the universe $(S_1 \oplus S_2) \oplus S_3$. Like $|\Psi\rangle$, this has no dependence on any time parameter — yet if we use the term "evolve" to mean "change to successive eigenvalues of $\hat{T}$", we can say the following about the dynamics of a universe governed by the Hamiltonian $\hat{\mathbf{H}}$:

System S_3 evolves independently under the Hamiltonian $\hat{h}$. Systems S_1 and S_2 evolve under the Hamiltonian $\hat{H}$ in all branches in which $\hat{T}$ has values between 0 and T, and do not evolve at all otherwise.

Now it is easily seen that under the approximations stated above (i.e. sums over t are replaced by integrals and α is very small), $|\Psi\rangle$ is a solution of the Schrödinger equation for a system with Hamiltonian $\hat{\mathbf{H}}$ — specifically it is an eigenstate of $\hat{\mathbf{H}}$ with eigenvalue zero. From (3), (8), (11) and (12),

$$\hat{\mathbf{H}}|\Psi\rangle = -i \sum_{x \in \mathrm{Sp}(\hat{X})} \lambda_x \int_{-\infty}^{\infty} \mu_t \frac{d}{dt}\left\{ \left[\theta(t<0)|x,\,0\rangle + \theta(0\leq t\leq T)e^{i\hat{H}t}|x,\,0\rangle + \theta(t>T)|x,\,x\rangle\right] |t\rangle \right\} dt. \quad (13)$$

Because of the properties of μ_t as $t \to \pm\infty$, the boundary term on integration by parts vanishes, so

$$\hat{\mathbf{H}}|\Psi\rangle = i \sum_{x \in \mathrm{Sp}(\hat{X})} \lambda_x \int_{-\infty}^{\infty} \frac{d\mu}{dt}\left[\theta(t<0)|x,\,0\rangle + \theta(0\leq t\leq T)e^{i\hat{H}t}|x,\,0\rangle + \theta(t>T)|x,\,x\rangle\right] |t\rangle dt. \quad (14)$$

Now, taking the kets $\{\,|t\rangle\}$ in (14) to be orthonormal (remember that the integral over t is really a sum over the values for which they *are* orthonormal) and using (7) and the orthonormality of the eigenstates of $\hat{X}$, we have

$$\left\|\hat{\mathbf{H}}|\Psi\rangle\right\|^2 = \int_{-\infty}^{\infty} \left|\frac{d\mu}{dt}\right|^2 dt = \alpha \to 0, \quad (15)$$

as stated.

There is one difference between this model and that of Page and Wooters. In their model the clock and the other subsystem were strictly non-interacting. In this model the "clock", which is S_3, the "rest of the universe", does interact with the other two systems (or rather, it acts on them and they do not react back) and plays a realistic role in the measurement process.