## Twistor regularisation of ultra-violet divergences

In **TN 29**, and in my **TMP** review, I suggested that the introduction of *inhomogeneous boundaries at infinity* into twistor diagrams should have the potential to eliminate ultra-violet divergences, while retaining a manifestly finite integral formalism. It's now possible to show this for a special case of such a divergence, namely the Feynman diagram

$$\text{i.e. } \int d^4x\, d^4y\; \phi_1(x)\phi_2(x)\left(\Delta_F(x-y)\right)^2 \phi_3(y)\phi_4(y) \qquad (1)$$

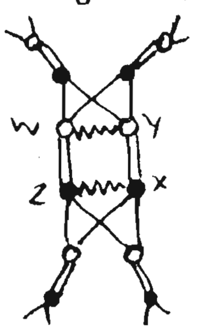
in massless $\phi^4$ theory. To do this I've gone back to the argument sketched out in TN 25 . This argument was basically on the right lines, but what I didn't see then was the essential role of *conformal symmetry breaking* in higher order Feynman diagram calculations - and this is the key factor.

In fact I should have noted that it's obvious that some such symmetry-breaking *must* come in. The regularisation of this divergent integral, as achieved by conventional QFT methods, is of the form

$$\log(p^2/\mu^2)$$

where p is the total ingoing (and outgoing) momentum and $\mu$ is some arbitrary mass. This doesn't just break conformal invariance: it's not even scale invariant. Note that although one may not think of $\phi^4$ theory as geunine physics, the integral being studied here is essentially the same as

in QED, and that the logarithmic factor in *that* context corresponds to the [zero-mass limit of the] Lamb shift - very well corroborated by experiment. So we should consider the logarithm as a genuine physical feature, making it imperative that some scale-breaking mechanism must be introduced. In fact it's not hard to write down a twistor diagram which does this and yields agreement with the logarithmic answer, namely




where the boundaries are on $\bar{Z}X = \mu$, $WY = \mu$, i.e. they are inhomogeneous boundaries at infinity, capable of breaking the scale invariance.

This is very encouraging, as it agrees with the general "skeleton" pattern postulated for the twistor version of Feynman diagrams. But can this diagram be *derived* - not just written down ***ad hoc*** using knowledge of the conventional regularised answer?

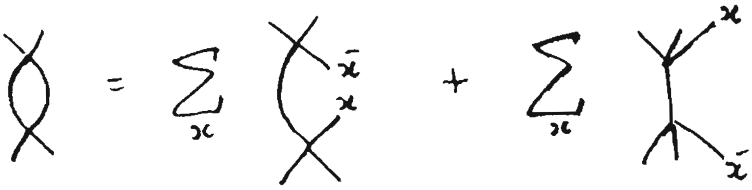
To analyse the problem, first note that in momentum space the Feynman integral appears as the (divergent) integral

$$\int \frac{d^4k}{k^2(p-k)^2}$$

where the integration is to be done according to the Feynman prescription. By elementary complex analysis, this prescription means that at least formally it is the same as

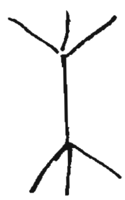
$$\int \frac{d^4k\ \delta^+(k^2)}{(p-k)^2} + \int \frac{d^4k\ \delta^+((p-k)^2)}{k^2}$$

Here the $\delta^+$ functions are just on shell propagators, which can be thought of as sums over a complete set of free states; thus we have

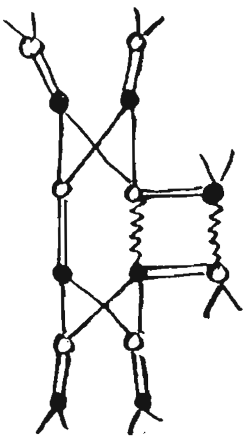



These sums are divergent, but the tree diagrams themselves are supposed to be finite, and the next thing is to study these tree diagrams in detail.

Much of this analysis has already been done in TN 25, and so I shall here simply assert that using the information described there, the $\phi^4$ diagram



is a finite, conformally invariant functional of the fields and can be represented exactly by the twistor diagram:



(2)

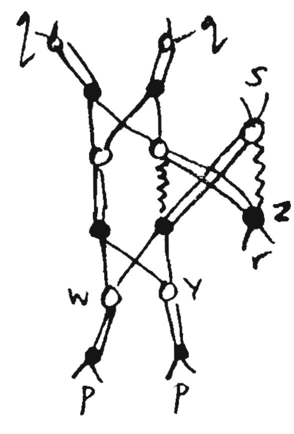
It's quite another story with the other channel. Let's be specific and use particular elementary states. Without loss of generality we can consider

$$\int \frac{d^4x\, d^4y\ \Delta_F(x-y)}{((x-p)^2)^2\,(x-s)^2\,((y-q)^2)^2\,(y-r)^2} \qquad (3)$$

where p, r are in the past tube and q, s in the future tube. This must yield a function F(p ,q ,r ,s ) satisfying

$$\left(\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p} + 2\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial s}\right) F(p,q,r,s) = \int \frac{d^4x}{((x-p)^2)^2\,(x-s)^2\,((x-q))^2\,(x-r)^2} \qquad (4)$$

The first difficulty is that the Feynman integral (3) is divergent, a problem swept under the carpet in the conventional approach where $1/k^2$ is called "finite" although it's singular at $k^2 = 0$. This means that we are driven first to find ***a regularisation for this tree diagram*** - a procedure quite unlike the conventional approach. In doing this we can be guided by the regularisation of the Møller scattering divergence. This naturally suggests the possibility that the divergence encountered here is regularised by the inhomogeneous twistor integral:



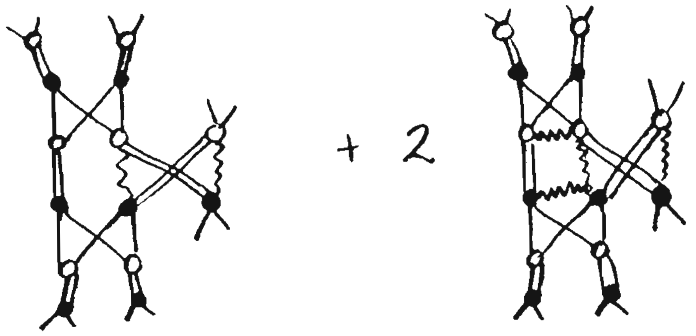

(5)

Calculation shows however that this doesn't satisfy the differential equation (2). In fact the two sides of the equation fail to match by [a multiple of] $\{((p-q)^2)^2(q-r)^2(p-s)^2\}^{-1}$

It follows that if we put

$$F(p,q,r,s) = \{(5)\} + 2\,\frac{\log((p-q)^2)}{(p-q)^2(q-r)^2(p-s)^2}$$

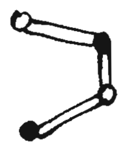
then this new F satisfies the essential equation (4). Note that a scale breaking element has entered now. It now turns out that this revised candidate for the regularised tree amplitude can be put in the form



(6)

where the extra term contains inhomogeneous boundaries at infinity to do the scale breaking. This looks very promising! It appears that we can now sum over the states as required, i.e. replace


by



The conformally invariant expressions cancel leaving just the contribution from the scale-breaking part. Unfortunately this leads to exactly TWICE the right answer (and so twice the right Lamb shift.) What's gone wrong? The trouble is that we haven't shown that (6) is a genuine regularisation of the divergent Feynman integral in (3); there could be other regularisations which differ by solutions of the homogeneous equation

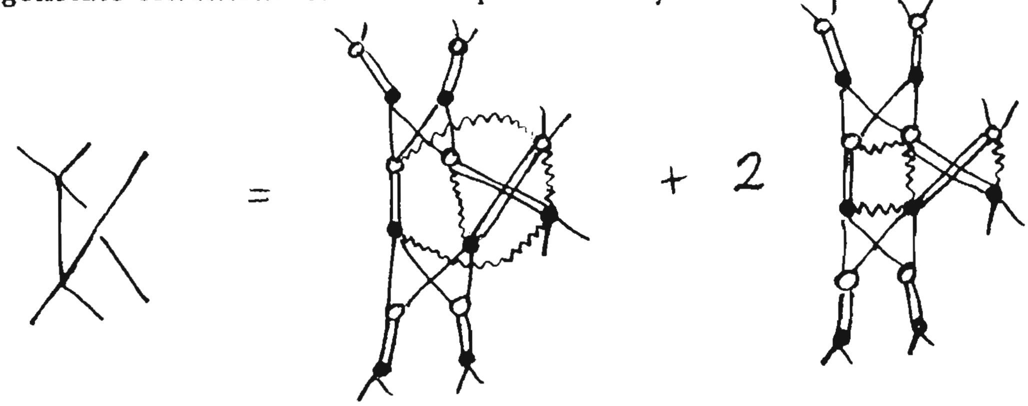
$$\left(\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p} + 2\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial s}\right) F(p,q,r,s) = 0$$

Indeed (6) is NOT a genuine regularisation. This is demonstrated by the fact that the interior of the diagram (5) doesn't satisfy the spin-0 eigenstate condition, i.e. that it's an eigenstate with eigenvalue 0 of

$$(Y.Z)\left(\frac{\partial}{\partial Y}\cdot\frac{\partial}{\partial Z}\right)$$

whilst the scale-breaking diagram added on in (6) does satisfy it. This means that the total functional of fields represented by (6) doesn't project out the spin-0 part of the fields meeting at a vertex - as it must to be a genuine regularisation of (3).

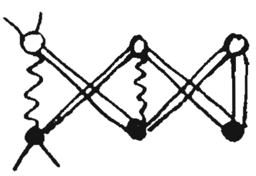
To cut a long story short, there DOES exist another completely finite functional of the fields which satisfies both (4) and the relevant spin eigenstate conditions. It can be represented by:




(7)

Note that scale-breaking inhomogeneous boundaries at infinity come into the integral thus introduced. The resulting total functional of fields is still not uniquely fixed by these conditions, so some further characterisation of satisfactory regularisation is still required. But I will assume that this is in fact the right answer for the tree diagrams.

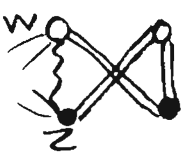
Now we can sum over states. The essential idea here is that these divergent sums are also regularised by twistor diagram inhomogeneity - but this time we only need a version of the "Møller" mechanism. For instance we can evaluate



as $\log\left(\frac{k\,K}{K\,k}\right)$

$= 0$

The basic reason for the finite answer is that in integrating out the states,




becomes $\displaystyle\oint_{W.Z>k} \frac{DWZ}{(W.Z)^2}\cdots$

so that the 'k' saves the pole from meeting the boundary. This mechanism can be applied consistently and this time we get the *right* answer for the loop integral, as there is now a partial cancellation between the scale-breaking terms.

I hope this analysis can be generalised to encompass all divergences (including vacuum diagrams) systematically, but much more work is needed yet.

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