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Preferred parameters on curves in conformal manifolds

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What follows are some observations made while considering whether one can construct analogues in conformal differential geometry of the 'pinched curvature, injectivity radii and minimising geodesics' ideas that lead (for example) to the Sphere Theorem in Riemannian differential geometry. These considerations are at a very early stage, but have led indirectly to an interesting (and as far as we know original) result about the distribution of curvature on a closed curve in $\mathbb{R}^n$.

Let γ be a *curve* (i.e. a smoothly immersed 1-dimensional submanifold) in an n-dimensional conformal Riemannian manifold. Choose a metric in which to work and parametrise γ by an arc-length parameter t and let U^a be the unit tangent vector. The *curvature* is given by $\kappa = \sqrt{A^b A_b}$, where $A^b = U^a \nabla_a U^b$.

As observed by Cartan (for an account in this language see the authors' paper in Proc. AMS **108** (1990), 215–221), such a curve has a natural *projective structure*—i.e. a family of preferred parameters related by fractional linear transformations under $SL(2, \mathbb{R})$. The function s on γ is a preferred parameter if it obeys the ('inhomogeneous Schwarzian') equation

$$(s')^{-1}s''' - \tfrac{3}{2}(s')^{-2}(s'')^2 = \tfrac{1}{2}\kappa^2 + P$$

where 'dash' denotes differentiation with respect to t and $P = P_{ab}U^aU^b$ where P_{ab} is the usual trace-modified multiple of the Ricci tensor. The simplified form of the equation when compared with the above reference is due to our use of an arc-length parameter.

In order to study the behaviour of these parameters we substitute $\xi = s(s')^{-1/2}$ since a brief calculation shows that these *ξ-parameters* obey the linear equation

$$\xi'' + (\tfrac{1}{4}\kappa^2 + \tfrac{1}{2}P)\xi = 0.$$

We define the *index* of a curve from A to B to be the number of zeroes (excluding the initial one) that the ξ-parameter with $\xi(A) = 0, \xi'(A) = 1$ has *before* reaching B. The location of, (and hence the number of) such zeroes is conformally invariant. As a first step towards investigating the properties of this index we have considered its behaviour on closed curves in $\mathbb{R}^n$. If γ is a (geometric) circle, it is easy to check that the first zero of any ξ-parameter occurs exactly at the starting point after one full traverse of the curve. As we see below, this characterises the circles among all closed curves.

We begin with a result (perhaps of interest in its own right) about the distribution of curvature on a closed curve.

Proposition *Let γ be a closed curve in $\mathbb{R}^n$ of length l, parametrised by arc-length t. Let κ be the curvature of γ. Then*

$$\int_0^l \kappa^2 \sin^2\left(\frac{\pi t}{l}\right) dt \geq \frac{2\pi^2}{l}$$

with equality if and only if γ is a circle.

The proposition is proved by expanding the coordinates as functions of t in Fourier series and performing some essentially trivial manipulations.

Proposition *Let γ be a closed 1-dimensional submanifold of $\mathbb{R}^n$ which is not a geometrical circle. Then any closed curve which consists of traversing γ once from some chosen starting point has index at least 1.*

Proof Let γ be of length l and consider the eigenvalue problem

$$\left(-\frac{d^2}{dt^2} - \tfrac{1}{4}\kappa^2\right)\xi = \lambda\xi, \qquad \xi(0) = \xi(l) = 0$$

on the interval $[0, l]$. Then the ξ-parameter with $\xi(0) = 0, \xi'(0) = 1$ will have a zero before $t = l$ if and only if zero is greater than the least eigenvalue of this problem. Since the operator on the left-hand side is bounded below we know that the least eigenvalue is always less than

$$\int_0^l \phi(t)\left(-\frac{d^2}{dt^2} - \tfrac{1}{4}\kappa^2\right)\phi(t)dt$$

for any function the integral of whose square over $[0, l]$ is unity.

Taking $\phi(t) = \sqrt{2/l}\sin^2(\pi t/l)$ we see that a sufficient condition is that

$$\int_0^l \kappa^2 \sin^2(\pi t/l)dt \geq \frac{2\pi^2}{l}.$$

The result then follows from Proposition 1. □

It is unclear at this stage whether these ideas (together perhaps with a study of the 'exponential map' for conformal circles) will lead to any interesting results on conformal manifolds. It would be interesting to know (for example) whether conformal circles are in any sense index-minimising curves. As a starting point however one can (easily) prove results such as:

Proposition *If M is a compact Riemannian conformal manifold such that there is a metric in the conformal class with pinched sectional curvatures*

$$\frac{2(n-1)}{4(n-2)k^2 + n} \leq K \leq 1$$

and $k \geq 1$ is an integer, then any two points can be joined by a curve of index less than k.

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