Self-dual manifolds need not be locally conformal to Einstein

There are global topological obstructions to the existence of Einstein metrics in four dimensions (which imply, for example, that $\mathbb{CP}_2 \# \mathbb{CP}_2 \# \mathbb{CP}_2 \# \mathbb{CP}_2$ does not admit such a metric). On the other hand, $\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2$ always admits a self-dual metric (as shown abstractly by Donaldson & Friedman [2] and explicitly by LeBrun [5]). The purpose of this note is to observe that there are also local obstructions to the existence of an Einstein metric within a given self-dual conformal class. We shall discuss the obstruction

$$K_{abc} = C^{efgh}C_{efgh}\nabla^d C_{abcd} - 4C^{efgh}C_{abch}\nabla^d C_{efgd}$$

of Kozameh, Newman, & Tod [4] where C_{abcd} is the Weyl tensor. This tensor is easily shown to be conformally invariant and the contracted Bianchi identity shows that it vanishes in the case of an Einstein metric. If C_{abcd} is self-dual, then it may be written as $K_{abc} = -4K_{A'B'C'C}\,\varepsilon_{AB}$ where

$$K_{A'B'C'C} = \tilde{\Psi}^{E'F'G'H'}\tilde{\Psi}_{E'F'G'H'}\nabla_C^{D'}\tilde{\Psi}_{A'B'C'D'} - 2\tilde{\Psi}^{E'F'G'H'}\tilde{\Psi}_{A'B'C'H'}\nabla_C^{D'}\tilde{\Psi}_{E'F'G'D'}.$$

In [1], Baston & Mason identified two tensors

$$E_{abc} = \Psi_{ABCD}\nabla^{DD'}\tilde{\Psi}_{A'B'C'D'} - \tilde{\Psi}_{A'B'C'D'}\nabla^{DD'}\Psi_{ABCD}$$
$$B_{ab} = (\nabla^C_{A'}\nabla^D_{B'} + \Phi^{CD}_{A'B'})\Psi_{ABCD}$$

whose vanishing in the case of algebraically general Weyl curvature is necessary and sufficient for the existence of an Einstein scale. These tensors evidently vanish for a self-dual metric. It is, therefore, enlightening to notice that

$$2\Psi_C{}^{EFG}\left(\delta^{E'}_{A'}\delta^{F'}_{B'}\delta^{G'}_{C'}\tilde{\Psi}^2 - 2\tilde{\Psi}^{E'F'G'H'}\tilde{\Psi}_{A'B'C'H'}\right)E_{efg} = \Psi^2 K_{A'B'C'C}$$

(where $X^2 = X^{ABCD}X_{ABCD}$). Thus, if E_{abc} vanishes and the Weyl curvature is algebraically general (whence, in particular, Ψ^2 is nowhere vanishing), then K_{abc} also vanishes.

We claim, however, that there are self-dual metrics for which K_{abc} is non-zero and hence a genuine obstruction to the existence of an Einstein scale. In fact, the metric

$$(dw + 2xy\,dy - x\,dz)^2 + 2y^2z\,(dx^2 + dy^2) + y^2dz^2$$

is self-dual but K_{abc} is nowhere vanishing — for example,

$$K \lrcorner \left(\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial x}\right) = \frac{9}{8y^7z^4} \quad \text{and} \quad K \lrcorner \left(\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial y}\right) = \frac{9x(y^2-2z)}{4y^8z^5}.$$

The explicit computation of K_{abc} is far from easy. These results were obtained via a computer program (written in "maple" and available upon request: it computes K_{abc} along with other differential geometric fauna for any explicit metric and can also check whether the metric is self-dual).

Of course, the metric above was constructed to be self-dual — notice that $\partial/\partial w$ is a Killing vector so the metric arises from an Einstein-Weyl space together with a generalized monopole as described by Jones & Tod [3]. In fact, the Einstein-Weyl space in question is hyperbolic 3-space. This is also the basis of LeBrun's explicit metrics [5] on $\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2$. The generalized monopoles that he employs, however, are derived from Green's functions for the hyperbolic Laplacian. Whilst extremely natural, they are computationally more difficult and we have not yet succeeded in completing the calculation of K_{abc} in this case. We suspect that it will turn out to be non-zero.

The tensor K_{abc} is evidently an obstruction to rescaling the metric so that $\hat{\nabla}^d \hat{C}_{abcd} = 0$, a so-called "C metric". Indeed, its derivation in [4] is from

$$\nabla^d C_{abcd} + \Upsilon^d C_{abcd} = 0, \qquad *$$

noting that if $C^2 \equiv C^{abcd}C_{abcd} \neq 0$, then $\Upsilon^h = -4\,C^{efgh}\nabla^d C_{efgd}/C^2$. Substituting back into $*$ gives K_{abc}/C^2. In the algebraically general case Υ^h is automatically closed [4]. We presume this is an extra condition in s-d case.

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