Families of invariants

There are many well known examples of linear conformally invariant differential operators: For a flat four dimensional conformal geometry the Laplacian $\Delta := \nabla^a\nabla_a$ is invariant when acting on densities of weight -1. If f is a density of weight 1 then

$$\nabla_{(a}\nabla_{b)_0} f \tag{1}$$

(where $()_0$ denotes the trace-free symmetric part) is invariant.

One can also construct non-linear invariants. For example, in four dimensions, if again f has weight 1 then

$$f\Delta f - 2\nabla^a f\nabla_a f \tag{2}$$

is invariant. It is interesting to note that this latter invariant is closely related to the Laplacian invariant via the identity

$$-f^3\Delta f^{-1} = f\Delta f - 2\nabla^a f\nabla_a f.$$

Indeed we can use this to deduce the invariance of (2) from the invariance of the Laplacian since f^{-1} has weight -1. More generally if f has weight w then $f^{2+1w}\Delta f^{-\frac{1}{w}}$ is clearly invariant. Expanding this out we obtain a family of invariants parametrised by weight w:

$$wf\Delta f - (w+1)\nabla^a f\nabla_a f.$$

The Laplacian, or at least $f\Delta f$, is seen to be just a special case of this family. By similar reasoning the invariant (1) is found to be a special case (if we ignore overall left multiplication by f) in the family

$$wf\nabla_{(a}\nabla_{b)_0} f - (w-1)\nabla_{(a} f\nabla_{b)_0} f.$$

Indeed given any invariant on (non-zero weight) densities, which is polynomial in jets of the density, one can use this technique to generate the family to which it belongs. In other words provided we avoid invariants of the functions[1] then all invariants on densities of any given weight can be

[1] In fact the reader will observe that putting $w = 0$ in the above formulae does yield an invariant. However at this stage it is not known how many invariants of functions do not arise in this fashion. $\nabla_a f^s$ is one example.

obtained in this manner from the set of all invariants on any other given weight.

For invariants of densities this procedure and these results also work in other dimensions and for other structures (for example projective geometries) and also for the corresponding curved cases. On the other hand it is difficult to imagine that *this* scheme for producing the families of invariants can be generalised to deal with quantities other than densities (i.e. weighted tensors and spinors). However, at least in the flat case, these families can be generated in other ways [1] that work equally well for tensors. It is likely that similar results hold for this more general case – i.e. that all invariants occur in families. If this is true then the problem of producing a complete theory of invariants is considerably reduced.

References

[1] A.R.G. A theory of invariants for flat conformal and projective structures. *preprint.*

A. Rod Gover

Quaternionic complexes.

R. J. Baston
Mathematical Institute
24–29 St. Giles
Oxford
OX1 3LB
U.K.

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Abstract

Each regular or semi-regular integral affine orbit of the Weyl group of $\mathrm{gl}(2n+2, \mathbb{C})$ invariantly determines a locally exact differential complex on a $4n$ dimensional quaternionic manifold.