A multiplicity one theorem for the Penrose transform

In the last TN, I outlined a method for computing the cohomology of certain sheaves on homogeneous twistor spaces Z for general complex semisimple Lie groups. I used the fact that (a) the sheaves in question are related to each other by certain explicit short exact sequences and (b) there is a basic sheaf (obtained by helicity raising from the canonical bundle) whose cohomology is simple to compute[1]. The only tricky point is to understand the maps in the resulting long exact sequences on cohomology. It turns out that there is an elegant inductive procedure for doing this in a good number of cases, including all of physical interest. The result may be viewed as an elegant extension of the Bott-Borel-Weil theorem to *open* homogeneous spaces. It also removes spectral sequences in the Penrose transform from recipes to their proofs.

Suppose we have a short sequence

$$0 \to A \to B \to C \to 0$$

obtained by helicity lowering and raising as in the last TN. Suppose we know $H^i(Z, A)$ and so $H^i(Z, B)$. To obtained $H^i(Z, C)$ from the long exact sequence

$$\cdots \to H^i(Z, A) \xrightarrow{\alpha} H^i(Z, B) \to H^i(Z, C) \to \cdots$$

we need to know α. These cohomology groups bear extra structure because $Z \subset G/R$ is a subset of a complex homogeneous space and A, B, C are homogeneous (see [2] for terminology etc.). They are all modules over the Lie algebra $\mathfrak{g}$ of G—in the usual four dimensional twistor picture, $G = \mathrm{SL}(4, \mathbf{C})$ covers the complex conformal group. We can therefore try to decompose $H^i(Z, A)$ etc. into $\mathfrak{g}$ irreducibles. If $\mathfrak{g}$ were the real Lie algebra of a compact Lie group the result would be a direct sum of finite dimensional vector spaces, each an irreducible representation of $\mathfrak{g}$. Since $\mathfrak{g}$ is complex, however, the best we can do is to write $H^i(Z, A)$ as a tower of submodules whose successive quotients are direct sums of (not necessarily finite dimensional) irreducible representations. We get a picture of $H^i(Z, A)$ as a building with each storey a home for a direct sum of irreducibles. Thus we might have

$$H^i(Z, A) \quad = \quad \begin{array}{|c|} \hline L_4 \oplus L_5 \\ \hline L_2 \oplus L_3 \\ \hline L_1 \\ \hline \end{array}$$

with L_1 a submodule. By requiring that A is irreducible, we can ensure that the L_i come from a known finite list of possibilities. The scheme is to figure out what α does to each L_i. The ideal answer is the following

[1]Similar ideas were suggested by Lionel Mason in an earlier TN.

Proposition *If L_i occurs in $H^i(Z, A)$ and in $H^i(Z, B)$ then α is an isomorphism between them.*

A much stronger result is in fact true. Recall that Z corresponds to a Stein open subset $X \subset G/P$ of a second homogeneous space. Suppose (G, P) and (G, Q) are both Hermitian symmetric pairs (this includes all cases of physical interest—see [2, chap 10]) and A is now any irreducible homogeneous sheaf on Z. Then

Theorem *An irreducible* g *module L occurs in at most one degree in any $H^i(Z, A)$ and then only once. There is an algorithm for determining when it occurs.*

This certainly implies the proposition, for if α is zero on an L occuring in both $H^i(Z, A)$ and $H^i(Z, B)$ then L occurs in $H^{i-1}(Z, C)$ and $H^i(Z, C)$. The algorithm follows from the long sequences.

The idea behind the proof is quite simple. There are certain special homogeneous bundles D called *singular* on G/R, distinguished by $H^i(G/R, D) = 0$ for all i. $\mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1)$ are all examples for ordinary twistor space in $\mathbf{CP}^3$. Indeed, any non singular bundle A can be sent to such a bundle by helicity lowering. The short exact sequences above are obtained by first doing this lowering and then raising back to the original helicity. Taking cohomology commutes with this helicity lowering. If L had multiplicity more than one in $H^i(Z, A)$, we could arrange to helicity lower A to a singular bundle D so that the helicity lowered L is not zero and still has multiplicity more than one in $H^i(Z, D)$. This means that proving the theorem reduces to proving it for such singular bundles.

A great deal is known about singular bundles in the Hermitian symmetric setting. It turns out [3] that they behave very much like *non singular* bundles for groups G', P', R' of the same kind as G, P, R *but smaller dimension!* The proof of the theorem now reduces checking that this behaviour extends to the Penrose transform, for that provides an inductive step on the rank (dimension) of G. Some rather beautiful subtleties emerge whilst doing this to explain how cohomology degrees are related between the pictures for G' and G. Details to come in [1].

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References

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