A SPINOR FORMULATION FOR HARMONIC MORPHISMS

by

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0. Introduction

The aim of this paper is to draw attention to the simple description of harmonic morphisms in terms of spinors, and to interpret the equations in terms of holomorphicity properties of sections of twistor bundles.

Harmonic morphisms have been studied by mathematicians for some time. They are defined as mappings between Riemannian manifolds which pull back germs of harmonic functions to germs of harmonic functions. Equivalently they are the harmonic mappings which are horizontally conformal (see [B1, B3, BW1] for details and further references). Thus if $\phi : M \to \mathbf{C}$ is a mapping from a Riemannian manifold M ($\dim M \geq 2$) with values in the complex plane $\mathbf{C}$, then ϕ is a harmonic morphism if and only if

$$g^{ab}\frac{\partial\phi}{\partial x^a}\frac{\partial\phi}{\partial x^b} = 0 \tag{0.1}$$

$$\Delta\phi = g^{ab}\left(\frac{\partial^2\phi}{\partial x^a\partial x^b} - \Gamma^c_{ab}\frac{\partial\phi}{\partial x^c}\right) = 0 \ , \tag{0.2}$$

where $g = g^{ab}$ is the metric on M and the Γ's are the corresponding Christoffel symbols. The first equation expressing horizontal conformality, the second harmonicity. In this note we concentrate on the case when $M \subset \mathbf{R}^4$ is an open subset of Euclidean 4-space. At the end we indicate the Euclidean $\mathbf{R}^3$ case and the Minkowski M^4 case.

To a harmonic morphism $\phi : M \to \mathbf{C}$, $M \subset \mathbf{R}^4$, we associate a pair of spinor fields $(\xi^A, \eta^{A'})$ defined on M. These satisfy the spinor equations

$$\begin{cases} \nabla_{AA'}\xi^A\eta^{B'} = 0 \\ \nabla_{AB'}\xi^C\eta^{B'} = 0 \ . \end{cases} \tag{1.7}$$

We interpret the projectivised fields $[\xi^A]$, $[\eta^{A'}]$ in terms of Gauss sections. The pair $([\xi^A], [\eta^{A'}])$ then determines a section of the well known twistor bundle $Z^+ \times Z^-$ over M (see [ES]). This ties in with the description of the second author [W] for submersive harmonic morphisms from a Riemannian 4-manifold to a surface. The equations (1.7) are then equivalent to holomorphicity equations for that section. It is worth pointing out that the spinor formulation in this note does not require the restriction that ϕ be submersive. This was necessary in [W] to guarantee a decomposition of the tangent space into well-defined vertical and horizontal spaces at each point. Thus we generalize here to arbitrary harmonic morphisms.

As an additional comment we note that a submersive harmonic morphism from a Riemannian m-dimensional manifold M to a surface is locally equivalent to an (m-2)-dimensional conformal foliation of M by minimal submanifolds. In the case when $m = 3$, we can remove the

(*) Supported by an S.E.R.C. Advanced Fellowship.

restriction 'submersive' and the foliation is by geodesics. Such conformal foliations are the Riemannian analogue of the well known shear-free null geodesic congruences, much studied by relativists in connection with zero rest mass fields (see [BW2, 3] for details).

In [BW1,2] all harmonic morphisms from a three-dimensional simply connected space form to a surface were determined. In these cases the harmonic morphisms are determined by pairs of holomorphic functions, see also Tod [T1, 2].

Throughout we use spinors as described in the Appendix of [PR, vol 2]. This enables us to consider spinors defined on a vector space with metric of arbitrary signature.

1. Harmonic morphisms from $\mathbf{R}^4$ in terms of spinors

We consider $\mathbf{R}^4$ with its standard Euclidean metric. Vectors x^a may be expressed in terms of spinors by the correspondence:

$$(x^0, x^1, x^2, x^3) \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} i\,x^0 + x^1 & x^2 + i\,x^3 \\ x^2 - i\,x^3 & ix^0 - x^1 \end{pmatrix} = x^{AA'}.$$

Writing $\partial_a = \partial/\partial x^a$, the spinor covariant derivatives $\nabla_{AA'}$ are given by

$$\nabla_{00'} = \frac{1}{\sqrt{2}}(-i\,\partial_0 + \partial_1)$$
$$\nabla_{01'} = \frac{1}{\sqrt{2}}(\partial_2 - i\,\partial_3)$$
$$\nabla_{10'} = \frac{1}{\sqrt{2}}(\partial_2 + i\,\partial_3)$$
$$\nabla_{11'} = \frac{1}{\sqrt{2}}(-\,i\,\partial_0 - \partial_1)\ .$$

Now let $M \subset \mathbf{R}^4$ be an open subset, and recall that $\phi : M \to \mathbf{C}$ is *horizontally conformal* if and only if

$$(1.1)\qquad \sum_a \left(\frac{\partial\phi}{\partial x^a}\right)^2 = 0\ ,$$

and ϕ is *harmonic* if and only if

$$(1.2)\qquad \sum_a \frac{\partial^2\phi}{(\partial x^a)^2} = 0\,.$$

So ϕ is a harmonic morphism if and only if (1.1) and (1.2) are satisfied. Let $\phi : M \to \mathbf{C}$ be a smooth mapping. From equation (1.1) we immediately deduce that ϕ is horizontally conformal if and only if

$$(1.3)\qquad \nabla_{AA'}\phi = \xi_A\eta_{A'}\ ,$$

for some spinor fields $\xi_A, \eta_{A'}$ defined on M.

Remarks 1) We always have the freedom $(\xi_A, \eta_{A'}) \to (\lambda\xi_A, (1/\lambda)\eta_{A'})\,, \lambda \in \mathbf{C}$.

2) At a critical point of ϕ, $\nabla_{AA'}\phi = 0$ and one of $\xi_A, \eta_{A'}$ is zero.

3) Equation (1.1) is the condition that the gradient be a complex null vector field. That is $\nabla\phi.\nabla\phi = 0$.

Since (1.2) is equivalent to $\nabla^{AA'}\nabla_{AA'}\phi = 0$, we have that, if ϕ is horizontally conformal, so that $\nabla_{AA'}\phi = \xi_A \eta_{A'}$ for spinors $\xi_A, \eta_{A'}$, then ϕ is harmonic if and only if

(1.4) $$\nabla^{AA'}\xi_A\eta_{A'} = 0 .$$

Conversely, given a pair of spinor fields $\xi_A, \eta_{A'}$ on M, we would like conditions which ensure they determine a harmonic morphism. Now the product $\xi_A\eta_{A'}$ determines a null vector field v_a. We require $\nabla_{[a}v_{b]}$ to be zero. This is computed to be equivalent to the six spinor equations:

(1.5) $$\begin{cases} \text{(i)} & \nabla_{11'}\xi^1\eta^{1'} - \nabla_{00'}\xi^0\eta^{0'} = 0 \\ \text{(ii)} & \nabla_{01'}\xi^0\eta^{1'} - \nabla_{10'}\xi^1\eta^{0'} = 0 \\ \text{(iii)} & \nabla_{A0'}\xi^A\eta^{1'} = 0 \\ \text{(iv)} & \nabla_{A1'}\xi^A\eta^{0'} = 0 \\ \text{(v)} & \nabla_{0B'}\xi^1\eta^{B'} = 0 \\ \text{(vi)} & \nabla_{1B'}\xi^0\eta^{B'} = 0 . \end{cases}$$

Combining (1.4) and (1.5) we obtain

(1.6) **Theorem** *There is a correspondence between (i) harmonic morphisms* $\phi : M \to \mathbf{C}$, $M \subset \mathbf{R}^4$, *and (ii) pairs of spinor fields* $(\xi^A, \eta^{A'})$ *on* M *satisfying the spinor equations:*

(1.7) $$\begin{cases} \nabla_{AA'}\xi^A\eta^{B'} = 0 \\ \nabla_{AB'}\xi^C\eta^{B'} = 0 . \end{cases}$$

Proof It is clear that (1.7) implies equations (1.4) and (1.5). Conversely, suppose we consider the first of equations (1.7) with A' = B' = 0. Then

$$\nabla_{00'}\xi^0\eta^{0'} + \nabla_{10'}\xi^1\eta^{0'} = (\nabla_{00'}\xi^0\eta^{0'} + \nabla_{11'}\xi^1\eta^{1'} + \nabla_{10'}\xi^1\eta^{0'} + \nabla_{01'}\xi^0\eta^{1'})/2 \quad \text{by (1.5) (i) and (ii)}$$
$$= 0 \quad \text{by (1.4).}$$

The other equations are proved similarly.

Remark In terms of the geometric description of [PR]. At each point $x \in M$ where $\nabla_{AA'}\phi \neq 0$, $\xi_A(x)$ determines an α-plane $\alpha(x)$ on the quadric $Q_2 \subset \mathbf{C}P^3$, and $\eta_{A'}(x)$ determines a β-plane $\beta(x)$. Then $\alpha(x), \beta(x)$ intersect in a point of Q_2. This point corresponds (under the identification of Q_2 with the Grassmannian of oriented 2-planes in $\mathbf{R}^4$) to a real 2-plane through the origin in $\mathbf{R}^4$. This plane is the vertical space at x (the tangent to the fibre of ϕ through x), translated to the origin.

2. Examples

Particular examples of harmonic morphisms $\phi : \mathbf{R}^4 \to \mathbf{C}$ are given by maps which are holomorphic with respect to one of the Kähler structures on $\mathbf{R}^4$. Each Kähler structure arises from the standard one obtained by identifying $\mathbf{R}^4 \approx \mathbf{C} \times \mathbf{C}$, and composing with an isometry.

Use coordinates (z, w) for $\mathbf{C} \times \mathbf{C}$, so that $z = x^0 + ix^1$, $w = x^2 + ix^3$. Then

$\phi : M \to C$, $M \subset \mathbf{R}^4$, is holomorphic if and only if

$$\frac{\partial \phi}{\partial \bar{z}} = \frac{\partial \phi}{\partial \bar{w}} = 0$$

if and only if

$$\nabla_{00'}\phi = \nabla_{10'}\phi = 0 .$$

Then clearly $\det \nabla_{AA'}\phi = 0$ and

$$\nabla_{AA'}\phi = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix} \begin{pmatrix} 0 & * \end{pmatrix}$$

so that $\eta_{B'} = (0 \;\; \lambda)$, for some $\lambda \in C$. Similarly ϕ is -holomorphic if and only if $\eta_{B'} = (\mu \;\; 0)$ for some $\mu \in C$. We now consider the effect of an isometry on the spinor decomposition of $\nabla_{AA'}\phi$.

There is a well known double cover $SU(2) \times SU(2) \to SO(4)$. Suppose that $\theta \in SO(4)$ and define $\tilde{\phi} = \phi \circ \theta$. Then $\nabla\tilde{\phi}(x) = \nabla\phi(\theta(x)) \circ \theta$. If $(A, B) \in SU(2) \times SU(2)$ covers θ, then the induced action on spinors is given by

$$\xi_A \eta_{A'} \to A\xi_A \eta_{A'} B^* ,$$

where $B^* = \bar{B}^T$, so that

$$(\xi_A, \eta_{A'}) \to (A\xi_A, \eta_{A'}B^*)$$

(i.e. $\tilde{\xi}_A(\theta(x)) = A(\xi_A(x))$ and $\tilde{\eta}_{A'}(\theta(x)) = (\eta_{A'}(x))B^*$). Note that under the equivalence $(\xi_A, \eta_{A'}) \sim (\lambda\xi_A, \eta_{A'}/\lambda)$, this is independant of the choice of (A, B) covering θ. In particular we see that $\phi : M \to C$ is $\pm$holomorphic with respect to a Kähler structure obtained from the standard one by an orientation preserving isometry if and only if $[\eta_{A'}] \in CP^1$ is constant. Similarly ϕ is $\pm$holomorphic with respect to a Kähler structure obtained by an orientation reversing isometry if and only if $[\xi_A] \in CP^1$ is constant. To summarize.

(2.1) Theorem *If $\phi : M \to C$, $M \subset \mathbf{R}^4$ open, is a harmonic morphism, then ϕ is $\pm$holomorphic with respect to one of the Kähler structures on $\mathbf{R}^4$ if and only if either $[\eta_{A'}]$ or $[\xi_A]$ is constant.*

Another class of examples are those which have totally geodesic fibres. These are classified in [BW1]. If $\phi : M \to C$, M open in $\mathbf{R}^4$, is a harmonic morphism with totally geodesic fibres, let N denote the leaf space of the fibres. Locally and in favourable circumstances globally, N can be given the structure of a smooth Riemann surface and ϕ is given implicitly by the equation

$$\alpha_0(\phi(x))x^0 + \alpha_1(\phi(x))x^1 + \alpha_2(\phi(x))x^2 + \alpha_3(\phi(x))x^3 = 1 ,$$

where $x = (x^0, x^1, x^2, x^3)$, $\alpha = (1/2h)(1 - f^2 - g^2, i(1 + f^2 + g^2), -2f, -2g)$ and $f, g, h : N \to C \cup \infty$ are meromorphic functions. In this case it is easily checked that ξ_A, $\eta_{A'}$ are given by

$$\xi_A = \frac{1}{\sqrt{\sqrt{2}\,(\alpha'.x)\; h}} \begin{pmatrix} f - ig \\ i \end{pmatrix}$$

$$\eta_{A'} = \frac{1}{\sqrt{\sqrt{2}\,(\alpha'.x)\ h}}\ (-\,if + g \qquad 1)$$

(here f, g and h are evaluated at $\pi(x)$, where π is projection onto N).

Note that in general neither of these are projectively constant and so the harmonic morphisms are not ±holomorphic. It is not known whether (1.7) has any solutions globally defined on $\mathbf{R}^4$ apart from those with $[\xi_A]$ or $[\eta_{A'}]$ constant, such solutions would define new harmonic morphisms from $\mathbf{R}^4$ to $\mathbf{C}$.

3. Interpretation in terms of twistor bundles

Here we relate our spinor description to the description given by the second author [W] in terms of twistor bundles, thus interpreting the equations (1.7) in terms of holomorphicity properties of Gauss sections. We briefly summarize the results of [W].

Let V be a 2-dimensional distribution in an oriented 4-dimensional Riemannian manifold M, and let H be the corresponding orthogonal 2-dimensional distribution. We may locally choose orientations for each V_x, H_x, $x \in M$, so that the combined orientation of $V_x \oplus H_x = T_xM$ is that of M. We then define almost complex structures J^V, J^H on each V_x, H_x to be rotation through $\pi/2$. Note that changing the orientation of V_x changes that of H_x and replaces (J^V, J^H) by $(-J^V, -J^H)$. All results below will be independant of this change, so that there is no loss of generality in assuming J^V, J^H are globally chosen.

The Gauss section of V, $\gamma : M \to G_2(TM)$ then maps into the Grassmannian of oriented 2-planes in TM. The almost complex structures J^V and J^H combine to give almost complex structures $J^1 = (J^V, J^H)$ and $J^2 = (-J^V, J^H)$ on each T_xM. Note that J^1 is compatible with the orientation, i.e. there exists an oriented basis of the form e_1, J^1e_1, e_2, J^1e_2, whereas J^2 is incompatible. Let Z^+ (resp. Z^-) be the fibre bundle over M whose fibre at x is all metric almost complex structures on T_xM which are compatible (resp. incompatible) with the orientation; these are the well-known twistor bundles of M [ES]. The distribution V defines section $\gamma^1 : M \to Z^+$ and $\gamma^2 : M \to Z^-$ by $\gamma^1(x) = J^1$, $\gamma^2(x) = J^2$ (where J^1, J^2 both act on T_xM). Note that if M is an open subset of Euclidean space $\mathbf{R}^4$, the twistor bundles are trivial $Z^{\pm} = M \times S^2$ and there is a well-known holomorphic bijection $G_2(\mathbf{R}^4) \approx S^2 \times S^2$.

Given a submersive harmonic morphism $M^4 \to$ surface, the tangent spaces to its fibres give an integrable, minimal and conformal distribution.

(3.1) **Theorem** [W] *Let* V *be a 2-dimensional distribution on a 4-dimensional Riemannian manifold* M. *Then* V *is integrable, minimal and conformal if and only if the section* $\gamma^1 : M \to Z^+$ *is holomorphic with respect to the almost complex structure* J^2 *on* M *and the section* $\gamma^2 : M \to Z^-$ *is -holomorphic with respect to the almost complex structure* J^1 *on* M.

Now let $\phi : M \to \mathbf{C}$ be a submersive harmonic morphism from an open subset M of $\mathbf{R}^4$. Then the tangent planes to the fibres determine a 2-dimensional distribution V on M. At each point x, V_x is given by

$$[\partial_3\phi, -\partial_2\phi, \partial_1\phi, -\partial_0\phi] \in Q_2 \subset \mathbf{CP}^3,$$

where $Q_2 \approx G_2(R^4)$ is the standard identification of the Grassmannian with the complex quadric. Then a direct computation verifies that $\gamma^1 = [\eta_{A'}]$ and $\gamma^2 = [\xi_A]$ in terms of the spinor decomposition $\nabla_{AA'}\phi = \xi_A\eta_{A'}$.

Write $W = \nabla\phi$, then

$$W^a \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}\begin{pmatrix} iW^0 + W^1 & W^2 + iW^3 \\ W^2 - iW^3 & iW^0 - W^1 \end{pmatrix} = \begin{pmatrix} \xi^0 \\ \xi^1 \end{pmatrix} (\eta^{0'} \quad \eta^{1'})$$

and at each point $x \in M$,

$$[W] = [-i(\xi^0\eta^{0'} + \xi^1\eta^{1'}), \xi^0\eta^{0'} - \xi^1\eta^{1'}, \xi^0\eta^{1'} + \xi^1\eta^{0'}, i(\xi^1\eta^{0'} - \xi^0\eta^{1'})] \in CP^3 .$$

But the standard identification $CP^1 \times CP^1 \to Q_2 \subset CP^3$ is given by

$$([\xi^0, \xi^1], [\eta^{0'}, \eta^{1'}]) \to [-i(\xi^0\eta^{0'} + \xi^1\eta^{1'}), \xi^0\eta^{0'} - \xi^1\eta^{1'}, \xi^0\eta^{1'} + \xi^1\eta^{0'}, i(\xi^1\eta^{0'} - \xi^0\eta^{1'})] .$$

Thus, identifying CP^1 with $C \cup \infty$ by stereographic projection $[\xi^0, \xi^1] \to \xi^0/\xi^1$ etc., we find

$$\gamma^1 = \frac{W^2 + iW^3}{iW^0 - W^1} = \frac{iW^0 + W^1}{W^2 - iW^3}, \quad \gamma^2 = \frac{W^2 - iW^3}{iW^0 - W^1} = \frac{iW^0 + W^1}{W^2 + iW^3} . \tag{3.2}$$

Writing the spinor equations (1.7) in terms of W, they take the form

$$(3.3)\quad \begin{cases} \text{(i)} & (-i\partial_0 + \partial_1)(iW^0 + W^1) + (\partial_2 + i\partial_3)(W^2 - iW^3) = 0 \\ \text{(ii)} & (-i\partial_0 + \partial_1)(W^2 + iW^3) + (\partial_2 + i\partial_3)(iW^0 - W^1) = 0 \\ \text{(iii)} & (\partial_2 - i\partial_3)(iW^0 + W_1) + (-i\partial_0 - \partial_1)(W^2 - iW^3) = 0 \\ \text{(iv)} & (\partial_2 - i\partial_3)(W^2 + iW^3) + (-i\partial_0 - \partial_1)(iW^0 - W^1) = 0 \end{cases}$$

$$\begin{cases} \text{(v)} & (-i\partial_0 + \partial_1)(iW^0 + W^1) + (\partial_2 - i\partial_3)(W^2 + iW^3) = 0 \\ \text{(vi)} & (-i\partial_0 + \partial_1)(W^2 - iW^3) + (\partial_2 - i\partial_3)(iW^0 - W^1) = 0 \\ \text{(vii)} & (\partial_2 + i\partial_3)(iW^0 + W^1) + (-i\partial_0 - \partial_1)(W^2 + iW^3) = 0 \\ \text{(viii)} & (\partial_2 + i\partial_3)(W^2 - iW^3) + (-i\partial_0 - \partial_1)(iW^0 - W^1) = 0 . \end{cases}$$

Remark Of course these are equivalent to $\nabla_a W^a = 0$ and $\nabla_{[a}W_{b]} = 0$, expressing harmonicity and integrability respectively.

In order to show that equations (3.3) imply the holomorphicity results of Theorem (3.1), we consider a point x and suppose without loss of generality that ∂_2, ∂_3 span V_x. then $W^2 + iW^3 = W^2 - iW^3 = 0$ at x. Since the fibres of ϕ are minimal [BE], we also have $\partial_2 W^2 + \partial_3 W^3 = 0$ at x. In particular at x

$$(\partial_2 - i\partial_3)(W^2 + iW^3) = (\partial_2 + i\partial_3)(W^2 - iW^3) = 0$$

by minimality and integrability of the fibres.

Consider $\gamma^1 = (W^2 + iW^3)/(iW^0 - W^1)$. Then at x

$$(-i\partial_0 - \partial_1)(iW^0 - W^1) = 0$$

by (3.3)(viii). By horizontal conformality

$$(W^0 + iW^1)(W^0 - iW^1) = -(W^2 + iW^3)(W^2 - iW^3) .$$

So at x, either $W^0 + iW^1 = 0$ or $W^0 - iW^1 = 0$. Suppose $W^0 - iW^1 = 0$, in which case $W^0 + iW^1 \neq 0$. Then

$$(W^0 - iW^1)(\partial_2 + i\partial_3)(W^0 + iW^1) + (W^0 + iW^1)(\partial_2 + i\partial_3)(W^0 - iW^1) = 0$$

at x, so that

$$(\partial_2 + i\partial_3)(iW^0 + W^1) = 0$$

at x . Now (3.3)(vii) implies

$$(-i\partial_0 - \partial_1)(W^2 + iW^3) = 0,$$

so that

$$(-i\partial_0 - \partial_1)\gamma^1 = 0$$

and γ^1 is horizontally -holomorphic.

Writing $\gamma^1 = (iW^0 + W^1)/(W^2 - iW^3)$, a similar computation shows that $(\partial_2 + i\partial_3)\gamma^1 = 0$ and γ^1 is vertically holomorphic. Similarly γ^2 is horizontally -holomorphic and vertically -holomorphic. If on the other hand $W^0 + iW^1 = 0$, then the holomorphicity conditions are reversed. We have therefore shown directly that the spinor equations (1.7) give the equations of Theorem (3.1).

Conversely given a 2-dimensional distribution which is conformal, then it determines a null vector field which can be described by spinor fields $\xi_A, \eta_{A'}$. If the corresponding Gauss maps satisfy the holomorphicity equations of Theorem (3.1), then by that theorem the distribution is integrable and minimal and the spinor fields $\xi_A, \eta_{A'}$ satisfy equations (1.7).

This gives an interpretation for the spinor fields and equations of Theorem (1.6). The advantage of Theorem (1.6) over Theorem (3.1) is that it is valid for arbitrary harmonic morphisms (i.e. those with critical points).

4. **Minkowski space**

We consider a map $\phi : U \to \mathbf{C}$, U open in Minkowski space M^4, satisfying the equations:

(4.1) $$(\partial_0\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2 = 0$$

(4.2) $$\partial_0{}^2\phi - \partial_1{}^2\phi - \partial_2{}^2\phi - \partial_3{}^2\phi = 0.$$

The spinor correspondence is given by

$$x^a \quad \leftrightarrow \quad x^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix}$$

Exactly as for the $\mathbf{R}^4$ case we obtain

(4.3) **Theorem** *There is a correspondence between*

(i) *mappings $\phi : U \to \mathbf{C}$ satisfying equations (4.1) and (4.2) and*

(ii) *pairs of spinor fields $(\xi^A, \eta^{A'})$ on U satisfying the spinor equations:*

$$\begin{cases} \nabla_{AA'}\xi^A\eta^{B'} = 0 \\ \nabla_{AB'}\xi^C\eta^{B'} = 0. \end{cases}$$

5. **The Euclidean $\mathbf{R}^3$ case**

We define the spinor correspondence by

$$x^a = (x^1, x^2, x^3) \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} x^2 + ix^3 & -x^1 \\ -x^1 & -x^2 + ix^3 \end{pmatrix} = x^{AB}$$

Define the differential operators D_{AB} by

$$D_{00} = (\partial_2 - i\partial_3)/\sqrt{2}$$

$$D_{01} = -\partial_1/\sqrt{2}$$
$$D_{10} = -\partial_1/\sqrt{2}$$
$$D_{11} = (-\partial_2 - i\partial_3)/\sqrt{2}\,.$$

These agree with [S], equation (14). Let $M \subset \mathbf{R}^3$ be an open subset and suppose $\phi : M \to \mathbf{C}$ is a smooth mapping. Then, as for the $\mathbf{R}^4$ case, ϕ being horizontally conformal is equivalent to nullity of $D_{AB}\phi$ which is equivalent to

(5.1) $$D_{AB}\phi = \xi_A\xi_B$$

for some spinor field ξ_A on M.

If ϕ is horizontally conformal, so that $D_{AB}\phi = \xi_A\xi_B$, then ϕ is harmonic if and only if

(5.2) $$D_{AB}\xi^A\xi^B = 0$$

(if and only if $\xi^B D_{AB}\xi^A = 0$).

Conversely, as in [S], given a spacial null vector field $v = \mu^A\mu^B$, then curl v is given by $-i\sqrt{2}D^{C(A}\mu^{B)}\mu_C$. Combining this with equation (5.2) we obtain

(5.3) **Theorem** *There is a correspondence between harmonic morphisms* $\phi : M \to \mathbf{C}$, M *open in* $\mathbf{R}^3$, *and spinor fields* ξ^A *on* M *satisfying*

(5.4) $$D_{AB}\xi^A\xi^C = 0\,.$$

Note: A spinor field $\psi^{AB} = \xi^A\xi^B$ satisfying (5.4) may be interpreted as a null, source free, time independant solution to Maxwell's equations. This is in fact clear by expressing $\nabla\phi \equiv E + iB$ in real and imaginary parts. Then horizontal conformality implies $E.B = 0$, curl E = curl B = 0 is automatic and harmonicity gives div E = div B = 0 .

Given a harmonic morphism $\phi : M \to \mathbf{C}$, M open in $\mathbf{R}^3$, we can associate a Gauss map $\gamma : M \to S^2$, given by $\gamma(x)$ = unit positive tangent to the fibre of ϕ through x (see [B2, BW1]). In fact γ extends smoothly across critical points [BW3]. Then it is easily checked that in the chart given by stereographic projection $S^2 \to \mathbf{C}\cup\infty$, γ is represented by ξ_0/ξ_1. The equation (5.4) now has the simple interpretation of (i) minimality of the fibres, and (ii) horizontal holomorphicity of the Gauss map γ [B2, W].

Harmonic morphisms from open subsets of Euclidean space $\mathbf{R}^3$ have been completely classified in [BW1]. In fact locally ϕ is given implicitly by an equation

$$\alpha_1(\phi(x))x^1 + \alpha_2(\phi(x))x^2 + \alpha_3(\phi(x))x^3 = 1\,,$$

where $\alpha = (1/2h)(1 - g^2, i(1 + g^2), -2g)$ and h, g are meromorphic functions on a certain Riemann surface N (the leaf space of the corresponding folitation). In this case the corresponding spinor field is seen to be

(5.5) $$\xi_A = \frac{1}{\sqrt{\sqrt{2}\,\alpha'.x}}\left(\frac{1}{\sqrt{h}} \quad \frac{g}{\sqrt{h}}\right),$$

where g and h are functions of $\pi(x)$ where π is projection onto the leaf space N . By a result in [BW1], the only harmonic morphisms defined globally on $\mathbf{R}^3$ with values in a Riemann surface are given by an orthogonal projection followed by a weakly conformal map. In this case after appropriate choices of coordinates, $N \approx \mathbf{C}$, g is constant and $h(z) = z$. In particular this is true if and only if $[\xi_A]$ is constant.

Remark There is an interesting connection between harmonic morphisms $\phi : M \to \mathbb{C}$, M open in $\mathbb{R}^3$, and solutions to the Bogomolny equations (magnetic monopoles). For both are classified in terms of holomorphic curves in the compex surface TS^2 [BW1, H]. For examples such as the axially symmetric solutions of Prasad and Rossi, the region of physical interest appears to be the envelope of the fibres of the harmonic morphism. These are precisely the points x where $\alpha'.x = 0$ [B2, BW1] and so correspond to the singularities of the spinor field given by (5.5).

References

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