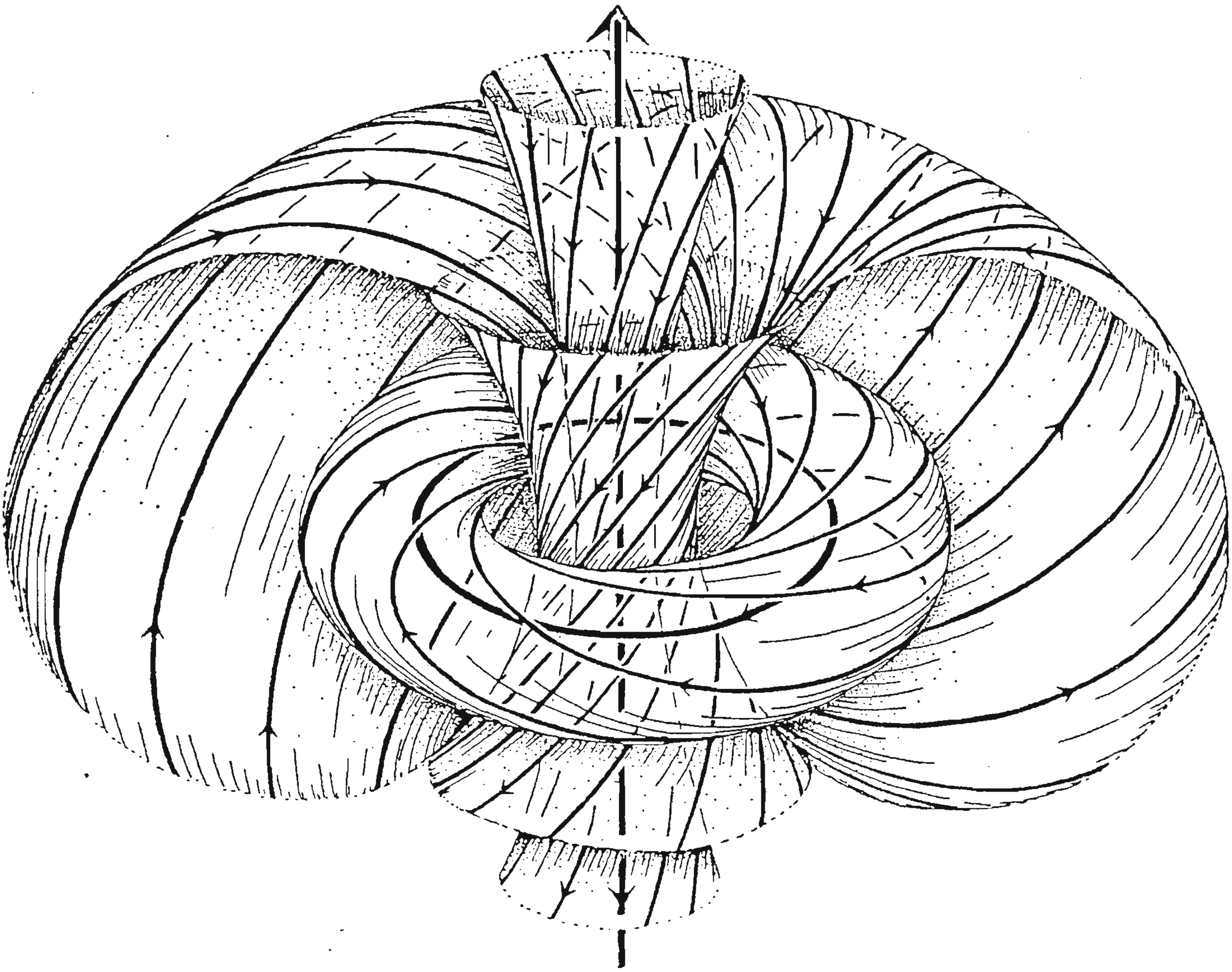
Twistor Newsletter (no 31: 29, October 1990)



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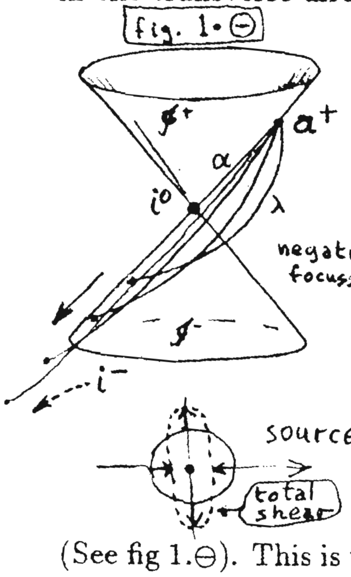
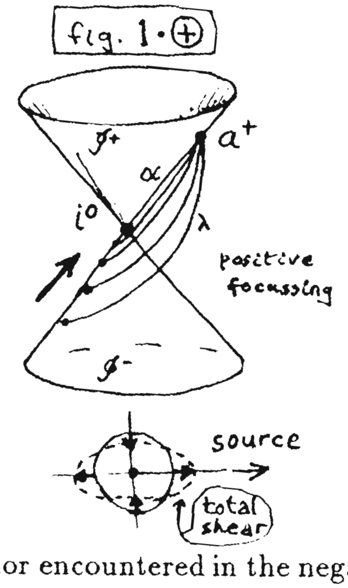
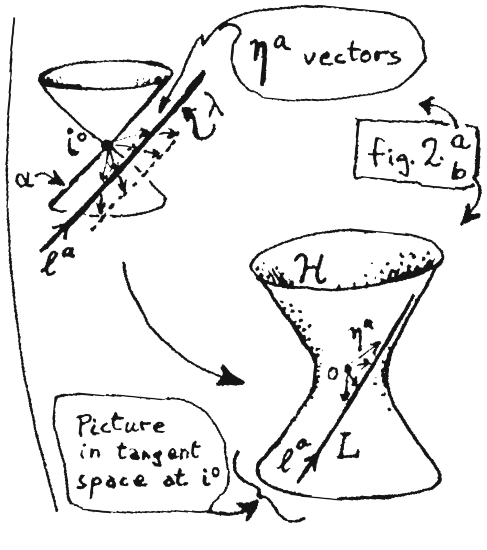
Mass Positivity from Focussing and the Structure of i^o

Recently, one of us (RP in **TN** 30, [1]) outlined an argument to show that in asymptotically simple space-times, satisfying a certain assumption, the mass is non-negative. The assumption –the *null conjugate point condition*– requires that every endless null geodesic contain a pair of conjugate points. This condition is physically reasonable. It is implied by completeness of null geodesics, a weak energy condition ($T_{ab}l^a l^b \geq 0$ if $l_a l^a = 0$) and genericity. The purpose of this contribution is to examine more closely what it is about the structure of spatial infinity, i^o , that the argument establishes. In particular, we will be able to establish that the mass in question is indeed the *ADM mass* at i^o . More precisely, our main result is that the ADM 4-momentum P_a of an asymptotically simple space-time satisfying the null conjugate point condition is necessarily a future-causal (or zero) 4-vector at i^o . (In this article, the assumption of asymptotic simplicity will include, in addition to asymptotic flatness at null infinity, that at spatial infinity. Thus, we assume that the space-time is AEFANSI in the sense of [2] and that the parity condition of [3] is satisfied at i^o .)

Let us begin by reviewing the result presented in [1]. Let $\mathcal{M}$ be an asymptotically simple space-time. Fix a point a^+ on the future null infinity $\mathscr{I}^+$ of $\mathcal{M}$. Let it lie on the generator α of $\mathscr{I}$. The past-directed light rays from a^+ will be said to *focus negatively* if they meet $\mathscr{I}^-$ in a family of points which recede indefinitely into the past (i.e., towards i^-) as these rays approach α. Let us suppose that this occurs. Then, if we examine the light rays neighbouring a given ray λ through a^+, as λ approaches α, the total shear of these rays along λ has the form of a convergence in the radial direction and a divergence in the transverse direction.







(See fig 1.⊖). This is the behavior encountered in the negative mass Schwarzschild solution. In the positive mass Schwarzschild space-time, the situation is just the opposite. Now, we encounter *positive focussing.* In this case, light rays from a^+ meet $\mathscr{I}^-$ in a family of points that approaches i^o as the rays approach α, there is a divergence of rays in the radial direction and convergence in the transverse direction. Thus, if we move in radially, the rays spray away from each other as i^o is approached, whereas transversally they pinch toward one another. (See fig 1.⊕.) Now, as the null geodesic λ gets closer to α, various non-linea terms die off and the shear is given by the integral, $\int_\lambda ds\, C_{abcd} l^a l^c \eta^b \eta^d$, evaluated along λ, where l^a is a parallelly propagated tangent to λ, s the corresponding affine parameter, and where η^a is the *radial* connecting vector. Thus, if negative focussing occurs, the integral would be positive and if positive focussing occurs, the integral would be negative. It was shown in [1] that, *if $\mathcal{M}$ satisfies the null conjugate point condition, negative focussing cannot occur.* Hence, in this case, the integral is necessarily non-positive for all geodesics λ originating at some point a^+ on $\mathcal{I}^+$ which are sufficiently close to the generator α of $\mathcal{I}$ on which a^+ lies. In this article, we show that this result in turn implies that the ADM 4-momentum is a (future pointing) causal vector. Thus, the intuition derived from the Schwarzschild solution is indeed valid more generally.

Let us recall the structure available at i^o. The asymptotic conditions of [2] imply that the tangent space at i^o is well-defined and carries a (universal) Minkowskian metric of signature $(-+++)$. Further, along any C^1 curve with tangent vector η^a at i^o, $\Omega^{\frac{1}{2}} C_{abcd}$ admits a limit at i^o, where C_{abcd} is the Weyl tensor of the unphysical metric g_{ab}. We can decompose the limit into its electric and magnetic parts using the unit space-like vectors η^a at i^o, and thus acquire two *symmetric, traceless* tensor fields $E_{ab}(\eta)$ and $B_{ab}(\eta)$ on the hyperboloid $\mathcal{H}$ of unit space-like vectors at i^o. (Note that, by their definition, the two fields are tangential to $\mathcal{H}$). Let us focus on the asymptotic electric field, E_{ab}. It satisfies the field equation $D_{[a}E_{b]c} = 0$ on $\mathcal{H}$, where D is the derivative operator compatible with the natural metric $h_{ab} = g^o{}_{ab} - \eta_a \eta_b$ on $\mathcal{H}$, where $g^o{}_{ab}$ is the Minkowski metric in the tangent space at i^o. The information about the ADM 4-momentum P_a is contained entirely in E_{ab}:

$$P_a V^a := -\tfrac{1}{8\pi} \oint_S dS^a\, E_{ab} V^b, \tag{1}$$

where V^a is any vector in the tangent space at i^o and S is any 2-sphere cross-section of $\mathcal{H}$. (The field equation and the trace-free property of E_{ab} imply that it is divergence-free while the projection of V^a into the hyperboloid (forced by its contraction with E_{ab} in the integrand) is a conformal Killing field on $\mathcal{H}$, whence the surface independence of (1).)

The field equation on E_{ab} also implies that it admits a scalar potential, E:

$$E_{ab} = D_a D_b E + E h_{ab} \tag{2.a}$$

In any given conformal completion, the potential E can in fact be constructed explicitly from the asymptotic (unphysical) Ricci tensor. The fact that E_{ab} is trace-free implies that E must satisfy the (tachyonic) massive scalar field equation:

$$D^a D_a E + 3E = 0 \tag{2.b}$$

on $\mathcal{H}$. In the Schwarzschild space-time with 4-momentum $P_a = m t_a$, with $t.t = -1$, E is given by:

$$E = m\, \frac{1 + 2(t.\eta)^2}{\sqrt{1 + (t.\eta)^2}} \tag{3}$$

More generally, in physically interesting situations, the asymptotic magnetic field B_{ab} vanishes on $\mathcal{H}$ and the leading terms in the *physical* metric are dictated entirely by E:

There exists a coordinate system in terms of which the physical metric has the asymptotic form:

$$d\hat{s}^2 = (1+\tfrac{E}{\rho})^2 d\rho^2 + \rho^2 \left(h^o{}_{\mathbf{ab}} + \tfrac{h^1{}_{\mathbf{ab}}}{\rho} + \tfrac{h^2{}_{\mathbf{ab}}}{\rho^2} + ...\right) d\phi^{\mathbf{a}} d\phi^{\mathbf{b}}, \tag{4}$$

with $h^1{}_{\mathbf{ab}} = Eh^o{}_{\mathbf{ab}}$. Here $h^o{}_{\mathbf{ab}}$ is the metric on the unit time-like hyperboloid in Minkowski space. Thus, $(\rho, \phi^{\mathbf{a}})$ should be thought of as "asymptotically hyperboloidal" coordinates.

It is easy to check that the space of solutions to the equation $D_a D_b \bar{E} + \bar{E} h_{ab} = 0$ is precisely 4-dimensional. The solutions are of the form $\bar{E} = K_a \eta^a$, where K_a is a fixed vector in the tangent space of i^o . Thus, there is some gauge freedom in the choice of the potential; we can add to the natural potential E of E_{ab} any $\bar{E}$ without changing the value of the field E_{ab}. This freedom is intertwined with the fact that if one uses only the asymptotic conditions of [2], there is some ambiguity in the conformal completion at i^o . Given a completion, one can obtain a four parameter family of inequivalent ones by logarithmic translations. In the physical space language, there are the transformations of the type:

$$x^{\mathbf{a}} \rightarrow x^{\mathbf{a}} + K^{\mathbf{a}} log \rho, \tag{5}$$

where $x^{\mathbf{a}}$ are asymptotically Cartesian coordinates, $\rho^2 = x^{\mathbf{a}} x_{\mathbf{a}}$ and where $K_{\mathbf{a}}$ are constants. The new completions are C^1 related at i^o . Therefore, i^o and the tangent spaces in the two completions are naturally identified. Under this identification, the field E_{ab} of one completion is mapped to that of the second. The potentials E, however, are not preserved. On $\mathcal{H}$ we have:

$$E \rightarrow E + K_a \eta^a \qquad \text{and} \qquad E_{ab} \rightarrow E_{ab} \tag{6}$$

Since the field is unaffected, so is the ADM 4-momentum (and, to next order, also angular momentum). Thus, the logarithmic translations may be thought of as "gauge" in this framework. In a large class of space-times, this gauge freedom can be eliminated. Suppose, as in [3], that the electric field E_{ab} is reflection symmetric on $\mathcal{H}$. Then, we can demand that the potential should also be reflection symmetric. (Note that E of the Schwarzschild solution (eq 3) automatically satisfies the condition. In Minkowski space the requirement singles out the potential $E = 0$.) This requirement selects a unique potential and hence removes the logarithmic ambiguity in the completion. As a part of the boundary conditions at i^o we assume that E_{ab} is reflection symmetric and work in a completion in which the potential E is also even under reflection.

With this formalism at hand, let us return to the implication of [1] quoted at the end of the first para. As the null geodesic λ approaches the generator α in the completed space-time (fig 2.a), the connecting vectors η^a become, in the limit, the position vectors of points on a null geodesic (straight line) L in $\mathcal{H}$, and the tangent vector l^a is now parallelly propagated w.r.t. h_{ab} (fig 2.b). Since the argument of [1] tells us that integral of $C_{abcd}\eta^a\eta^c l^b l^d$ along λ is non-positive, in the limit we conclude that $\int_L ds\, E_{ab}l^a l^b \leq 0$. Using the expression (2) of E_{ab}, we have $E_{ab}l^a l^b = (l^a D_a)^2 E$, so that the last condition becomes:

$$\dot{E}^+ - \dot{E}^- \leq 0, \tag{7.a}$$

where $\dot{E} = l^a D_a E$ and $\pm$ denote, respectively, the values at the future and past (ideal) end points of L. Now, since E is reflection symmetric, the two terms on the left side of (6) add and we have:

$$\dot{E}^+ \leq 0. \tag{7.b}$$

To see the implication of this condition, let us examine the asymptotic form of E. Let us foliate the tangent space of i^o by a family of planes $t = const$ (with $t = -t^a\eta_a$, with t^a unit future-timelike) and consider the corresponding foliation of $\mathcal{H}$. Assuming that E admits a power series expansion of the type $\sum E^{(n)}(\theta,\phi)\, t^{-n}$, where n runs from some *finite* negative value to $+\infty$, the field equation implies that E must admit an asymptotic expansion of the following type:

$$E(t,\theta,\phi) = (a_0 + a_m Y_{1m})(\theta,\phi))t + \frac{a_m Y_{1m}(\theta,\phi)}{2t} + \frac{E^{(3)}(\theta,\phi)}{t^3} + ... \tag{8}$$

where $Y_{1m}(\theta,\phi)$ are the three $\ell = 1$ spherical harmonics. (We believe the required assumption is always satisfied in the reflection-symmetric case.) The condition $\dot{E}^+ \leq 0$ implies that the coefficient of the first term is non-negative and hence the 4-vector $\bar{P}_a = a_0 t^a + V^a$ at i^o –with t^a the above unit time-like vector orthogonal to the slices and the spatial vector V^a in the $t = 0$ slice, given by $V^a\eta_a = a_m Y_{1m}(\theta,\phi)$– is future-directed and causal. At first, we were misled into thinking that the coefficient of the first term is the mass-aspect at the future end of $\mathcal{H}$ and therefore the argument would show that the mass-aspect should be positive. This is incorrect. In fact, the electric field E_{ab} constructed from the leading order term (via eq (2.a)) vanishes identically whence the term makes no contribution whatsoever to the ADM 4-momentum integral. Rather, the mass-aspect is the third term, $E^{(3)}$ in the expansion. However, because we have restricted ourselves to even potentials E, it *does* follow that the ADM 4-momentum P_a constructed from the correct mass aspect is precisely given by the vector $\bar{P}_a$, which resides in the leading order term. Therefore, although we cannot conclude that the mass-aspect should be positive, (7.b) does indeed imply that the ADM-4-momentum is a causal, future-directed vector.

We conclude with two remarks. First, one can show that the vector space obtained by superposing the asymptotic mass-aspects of Schwarzschild solutions (3) (whose the 4-momentum is not restricted to be time-like) is dense in the space of all asymptotic mass-aspects $E^{(3)}(\theta,\phi)$ arising from smooth solutions to (2.b). We believe, furthermore, that the same is true of the entire solutions E everywhere on $\mathcal{H}$. Thus, in the reflection-symmetric case, one can work with superpositions of (3) without loss of generality. The second remark has to do with the leading term in the asymptotic expansion (8). The fact that this term diverges is an indication that there is a mis-match at i^o between the following two limits: sliding down a generator of $\mathscr{I}^+$, and, approaching i^o along a space-like direction and then making an infinite boost. The mis-match is a measure of the ADM 4-momentum. There is a similar mis-match in the way $\mathscr{I}^-$ is attached to i^o . This mis-match should show up in the metric coefficients rather than the curvature. If one allows E to acquire a non-symmetric part under reflection, one can, by a logarithmic translation, remove the mis-match between i^o and $\mathscr{I}^+$. However, then the mis-match with $\mathscr{I}^-$ is twice as big. There is something "cohomological" here: with one coordinate choice near i^o , it appears that $\mathscr{I}^+$ matches "smoothly" on to i^o , whereas for another coordinate choice near i^o it would be $\mathscr{I}^-$ that matches smoothly to i^o . The 4-momentum represents the mismatch between these two attempts at a smooth structure at i^o . It would seem that these two choices correspond to whether we use the intersections with $\mathscr{I}^+$ or with $\mathscr{I}^-$ to represent light rays in the space-time. There appears to be a relation to twistor theory here. A clearer treatment of this issue is needed to make further progress in a proper understanding of asymptotically flat space-times, e.g. along the lines initiated by Friedrich.

We have presented here only the overall picture. Some details are yet to be worked out fully. Also, the results can probably be generalized in a number of ways. A more complete account will appear elsewhere.

Abhay Ashtekar & Roger Penrose

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2. Abhay Ashtekar, In: General Relativity and Gravitation, Vol 2, edited by A. Held (Plenum, New York, 1980).
3. Abhay Ashtekar, Found. Phys. **15**, 419 (1985).

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Conformally Invariant Operators: Singular cases.

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Abstract

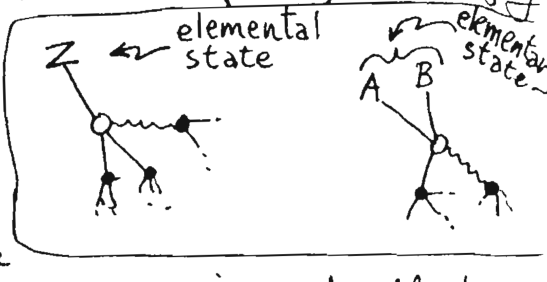
All invariant linear differential operators between bundles of singular weight on flat conformal manifolds are determined and shown to have analogues on general conformal manifolds, obtained by adding suitable curvature correction terms.

Twistor Theory for Vacuum Space-Times: a New Approach

A fundamental problem that has confronted the twistor programme since its earliest days has been to find the appropriate twistor theory for curved space-times, and most particularly vacuum space-times. Many of the stumbling blocks to different parts of the general programme seem to be rooted in the absence of an adequate union between the ideas of general relativity and twistor theory. (See R.P. in Bailey & Baston T.I.M.P. for a general survey of these problems and attempts at solution.) For some while it has seemed to me that one needs a concept of twistor with an intensely non-local space-time description. I shall present here a new concept of twistor that is, indeed, very non-local.

A difficulty with the asymptotic twistor/googly approach (with twistors defined at $\mathscr{I}$ only) has always been that twistors seem to have very little contact with the finite regions of space-time and with actual space-time points where, after all, the imposition of Einstein's vacuum equations $\mathbf{R}_{ab}=0$ is playing its role. The present suggestion may be regarded as a way of providing such contact.

The basic new ingredient is A.P.H.'s concept of elemental state (A.P.H. in T N 22 and p. 308-10 of F.A.T.T., Mason & Hughston) that, roughly speaking, is the "one-eared" version of the normal "two-eared" elementary state of standard twistor diagram theory. The two ears of the elementary states represent the two singularities which give the twistor function its H^1 character. That is necessary in order that the $\oint$ can be performed in order to provide the corresponding space-time field $\varphi_{\dots}$. This field is an H^0, i.e. it has a local space-time interpretation. The appropriate "$\oint$" on a "twistor function" with a single singularity (one ear) provides a cohomological space-time field. This is how the elemental states are described in space-time terms. There will be some covering (Stein) $\{\mathcal{U}_i\}$ of the relevant space-time region (say the complement of the α-plane represented by Z, in $\mathbb{C}\mathbb{M}^\#$), and a field $\varphi_{\dots}$ in each $\mathcal{U}_i$, providing a representative function of the cohomological object in question.



The idea here is to try to represent a twistor Z^α in terms of the (dual) "twistor function"

$$f(W_\alpha) = (Z^\alpha W_\alpha)_{-1}$$ (i.e. )

or perhaps, making use of A.P.H.'s insights with regard to twistor diagrams, something like

$$f(W_\alpha) = (Z^\alpha W_\alpha + k)_{-1} \, .$$

Here $(x)_{-1}$ is the <u>bracket factor</u> satisfying

$$\frac{d^2}{dx^2}(x)_{-1} = \frac{1}{x}$$

so we can take $(x)_{-1} = x \log x - x$ (or $x(\log x + \gamma - 1)$, where γ is Euler's constant — which may be more in line with A.P.H.'s prescriptions), or else $(x)_{-1} = x$ together with some rule that the contour has boundary on $x = 0$ (or $x + k = 0$), or else use a limiting argument with complex powers. If we try to integrate out the W_α-dependence to get a space-time "field", according to

$$\varphi_{....} = \oint (Z^\alpha W_\alpha (+k))_{-1} \, d^2 W$$

we end up with a <u>cohomological</u> field $\varphi_{....}$.

If f had been an ordinary (H^1-type) twistor function of homogeneity $n = 1$, then it would represent a helicity $s = \frac{3}{2}$ massless field ($n = 2s - 2$, for dual twistor functions), but since our f has the wrong cohomological structure for this, it represents a <u>cohomological</u> (or worse!) helicity $\frac{3}{2}$ field. The special feature of helicity $3/2$ for our purposes here is that the equations work (Buchdahl) also in Einstein vacuum space-times ($R_{ab} = 0$) — but only if we represent the field appropriately by a <u>potential</u>, which I shall take in the form

$$\boxed{\nabla_B^{B'} \gamma_{AB'C'} = 0, \qquad \gamma_{AB'C'} = \gamma_{AC'B'}}$$

where we take

$$\boxed{\gamma_{AB'C'} \text{ modulo } \nabla_{AB'} \nu_{C'} \text{ with } \nabla_A^{B'} \nu_{B'} = 0.}$$

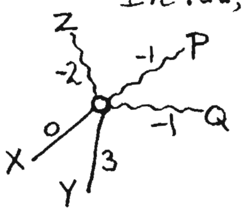
The consistency of these equations when $R_{ab} = 0$ is important to supergravity theorists, although the equations they use amount to the above without the symmetry condition on $\gamma_{...}$ and without the neutrino equation on $\nu_.$. The two forms are locally equivalent (cf. Penrose & Rindler S&S-T, Vol. 1, p. 370). An immediate (Buchdahl) consistency condition on the field equation for $\gamma_{...}$ is $R_{ab} = \lambda g_{ab}$, while we require $R_{ab} = 0$ in order that $\nu_{C'}$ act as a gauge.

The use of these spin $3/2$ equations here is totally different from their use in supergravity theory, however. The proposal here is to represent each twistor Z^α by an appropriate cohomology class (or "worse"?) on a vacuum space-time $\mathcal{M}$, the representative functions being helicity $\frac{3}{2}$ fields as described above. Note that such a representation is <u>doubly</u> <u>non-local</u>, firstly because of the cohomology and secondly because the field must be represented as a potential modulo gauge. In flat space-time $\mathbb{M}$, we would have a local field quantity given by

$$\varphi_{A'B'C'} = \nabla^A_{A'}\gamma_{AB'C'}\,, \quad \text{satisfying } \nabla^{A'}_A\varphi_{A'B'C'} = 0\,,$$

but in curved space-time $\mathcal{M}$, neither does the field equation for $\varphi_{\dots}$ follow nor is $\varphi_{\dots}$ independent of ν.

In $\mathbb{M}$, we can find such representative $\varphi_{\dots}$s and $\gamma_{\dots}$s by appealing to a twistor diagram. Here $X^\alpha = (ix^{AA'}\xi_{A'}, \xi_{A'})$ and $Y^\alpha = (ix^{AA'}\eta_{A'}, \eta_{A'})$ determine the field point (the line XY in $\mathbb{PT}$) and a local frame $\xi_{A'}$, $\eta_{A'}$ at that point; P^α and Q^α provide boundaries for the $\oint$ which determine the cohomological freedom.



It appears to be the case that the field $\varphi_{\dots}$ represents a <u>relative H^2</u>, which is taken relative to the α-plane determined by Z^α. The "cohomological status" of $\gamma_{\dots}$ is more obscure, because of the presence of logarithmic branching in its domain (work in progress!). In $\mathcal{M}$, where α-planes do not generally exist, in place of an α-plane, we seem to require a "smeared out" or "thickened" α-plane.

To see how this works explicitly in $\mathbb{M}$, take P and Q at infinity, i.e. of the form $P^\alpha = (\rho^A, 0)$, $Q^\alpha = (\sigma^A, 0)$. Evaluating the above twistor diagram in $\mathbb{CM}$, we get a field (proportional to)

$$\varphi_{A'B'C'} = \frac{\pi_{A'}\pi_{B'}\pi_{C'}}{\sigma\cdot\omega(x)\ \rho\cdot\omega(x)} \qquad (\times\, \xi^{A'}\xi^{B'}\xi^{C'})$$

where ω^A and $\pi_{A'}$ are the spinor parts of Z^α at the origin O of $\mathbb{CM}$, with

$$\omega^A(x) = \omega^A - ix^{AA'}\pi_{A'}\,,$$

and $\sigma\cdot\omega(x) = \sigma_A\omega^A(x)$, etc. A potential for $\varphi_{A'B'C'}$ is given by

$$\gamma_{AB'C'} = \frac{\pi_{B'}\pi_{C'}\sigma_A \log(\rho\cdot\omega(x))}{\sigma\cdot\omega(x)}\,.$$

Of course, in $\mathcal{M}$, we cannot expect to find representatives of this form, and there is a problem as to how one should characterize those particular cohomological elements that represent a <u>twistor</u> (elemental state), rather than something more general (i.e. Z rather than $f(Z)$). The best I can do so far is to go to $\mathbb{C}\mathscr{I}^+$ (or $\mathbb{C}\mathscr{I}^-$) and demand that it relate to an α-line in the same way as the above expressions in $\mathbb{CM}$. However it would be better if we can avoid direct reference to $\mathbb{C}\mathscr{I}$, if possible. (Work in progress.)

Note that if we had a "twistor" $Z^\bullet$ represented by $\gamma_{AB'C'}$ and a "dual twistor" $W_\bullet$ represented by $\zeta_{A'BC}$, then we might be able to define their scalar product by appropriately integrating (over some Pocchammer-type contour) the 3-form

$$\Delta = \zeta_{F'}{}^{ED}\gamma_E{}^{F'D'}e_{abcDD'}\,dx^a\wedge dx^b\wedge dx^c$$

(which is closed, and exact if either $\gamma_{\dots}$ or $\zeta_{\dots}$ is pure gauge. A.P.H. gives arguments to show that we [illegible] expect something like $f\log(W\cdot Z/t) + \chi - 1]\,W\cdot Z$ rather than just $W\cdot Z$

A measurement process in a stationary quantum system

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Conventionally, time appears in quantum mechanics as a c-number parameter on which physical quantities such as observables or states depend, but it is not itself an observable. This relic of classical physics must of course be regarded as a stopgap. Eventually quantum gravity will give us a unified theory of space-time and matter, from which in principle all dependence of observables on non-observables can be eliminated.

Some models that have this property have already been constructed, notably by DeWitt[1] as a way of making sense of canonical quantum gravity, and in a more hand-waving but more accessible way by Page and Wooters[2] considering simple systems with quantum clocks. All such models have in common a beautiful feature that is necessary but at first rather counter-intuitive: *The universe as a whole is at rest.* That is, the quantum state $|\Psi\rangle$ of the universe as a whole is an eigenstate of its Hamiltonian $\hat{\mathrm{H}}$. The reason why that is necessary is of course that otherwise the Schrödinger equation would give $|\Psi\rangle$, and therefore physical quantities, a dependence on an unphysical parameter t.

The observed phenomenon of quantities "changing with time" has nothing to do with any t-dependence. It is a correlation phenomenon. Although (or rather *because*) the universe is in an eigenstate of the Hamiltonian, it is not in an eigenstate of the position of hands on clocks, or of any other observable that the inhabitants might measure to tell them the time. Therefore it is in a superposition of such eigenstates, whose eigenvalues are the readings of clocks at different instants. Under the Everett interpretation this means that different instants co-exist. The division of the world into "instants of time" is just a special case of its division into Everett branches (often, somewhat misleadingly, called "parallel universes").

It seems to me that one of the most important types of time-dependence that we need to understand, both technically and physically, is that which occurs in measurement. Therefore I have tried to construct a model of a measurement process in a stationary universe. What follows is entirely heuristic. In other words, I am not trying to prove anything, only to answer the question, if time really is a quantum correlation phenomenon as just outlined, what might the state and Hamiltonian look like, and how would it all work out?

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1 DeWitt, B.S. *Phys. Rev.* **160** 1967.

2 Page, D.M. and Wooters, W. *Phys. Rev.* D27 1983.

As usual in the theory of measurement, let us divide the universe into three quantum subsystems:

(1) A system S_1, initially in state $|\psi\rangle$, in which the observable $\hat{X}$ with spectrum $\mathbf{Sp}(\hat{X})$ is to be measured. (What "initially" means will emerge below).

(2) An apparatus S_2, initially uncorrelated with S_1 and in a receptive state $|0\rangle$. S_2 has an observable $\hat{A}$, with $\mathbf{Sp}(\hat{X}) \subseteq \mathbf{Sp}(\hat{A})$, in which the measured value of $\hat{X}$ is to be stored.

(3) The rest of the universe, S_3.

Conventionally, the measurement would be described using a time-dependent Hamiltonian $\hat{H}(t)$ specifying an interaction between S_1 and S_2 with support only during the period $0 \leq t \leq T$ (say), and not involving S_3. If the measurement were perfect it would have the following effect during that period:

$$|x, 0\rangle \Rightarrow |x, x\rangle \qquad \left(\forall x \in \mathbf{Sp}(\hat{X})\right) \tag{1}$$

where the kets on the left and right of the "evolves-into" symbol "$\Rightarrow$" in (1) are the joint state of S_1 and S_2 immediately preceding and immediately following the measurement interaction, i.e. at times 0 and T respectively. The representation in (1) is in terms of simultaneous eigenstates of $\hat{X}$ and $\hat{A}$, labelled by the corresponding eigenvalues, and we are assuming for convenience that the receptive state of S_2 is the eigenstate $|0\rangle$ of $\hat{A}$.

If (as a further harmless idealization) $\hat{H}(t) = \hat{H}$, a constant operator, during the measurement,

$$|x, x\rangle = e^{i\hat{H}T}|x, 0\rangle \qquad \left(\forall x \in \mathbf{Sp}(\hat{X})\right). \tag{2}$$

That a Hamiltonian with this property exists follows from the unitarity of the required evolution (1).

The above description of a measurement process is incomplete in that it does not model time explicitly. The notions "before the measurement", "during the measurement" and "after the measurement", as well as "the duration of the measurement" are integral to the description and are all referred to as if they were observable quantities, but no quantum observable corresponding to any of these quantities is described — in fact there is no such observable in systems S_1 and S_2. Moreover system S_3 is described as not participating in the measurement process, but it is implicitly required that something outside S_1 and S_2 "switch the interaction on and off" at times 0 and T, to induce the necessary time-dependence in the dynamics of S_1 and S_2.

Now we follow Page and Wooters and extend the model to include time as an observable. Let $\hat{h}$ be the Hamiltonian of S_3, the "rest of the world", let $|0\rangle$ be some state of S_3 and define

$$|t\rangle \equiv e^{i\hat{h}t}|0\rangle \tag{3}$$

for all real t. Let $\mathcal{T}$ be a maximal set of real numbers such that the corresponding kets $|t\rangle$ are orthonormal for all $t \in \mathcal{T}$. If $|0\rangle$ and $\hat{h}$ have suitable properties, $\mathcal{T}$ will be a large set, approximating the real line in an appropriate physical sense, and there will exist an observable $\hat{T}$ of S_3 with $\mathbf{Sp}(\hat{T}) = \mathcal{T}$,

$$\hat{T} \equiv \sum_{t \in \mathcal{T}} t|t\rangle\langle t|, \tag{4}$$

which, as shall see, can serve as a time observable. That there can exist observables $\hat{T}$ and $\hat{h}$ with the properties just described may be shown by explicit construction: Given any set $\mathcal{T}$ of successive real numbers separated by intervals ε and any set $\{|t\rangle\}$ of orthonormal states of S_3 labelled by, among other things, the elements of $\mathcal{T}$, the observable

$$-\frac{i}{\varepsilon} log\left(\sum_{t \in \mathcal{T}} |t+\varepsilon\rangle\langle t| \right) \tag{5}$$

would serve as $\hat{h}$.

Suppose that S_1 starts in an arbitrary state $|\psi\rangle$ uncorrelated with S_2 or S_3.

$$|\psi\rangle = \sum_{x \in \mathbf{Sp}(\hat{X})} \lambda_x |x\rangle, \tag{6}$$

where

$$\sum_{x \in \mathbf{Sp}(\hat{X})} |\lambda_x|^2 = 1. \tag{7}$$

If the S_3-observable $\hat{T}$ as defined in (4) were really the "time" for systems S_1 and S_2, we should expect the universe as a whole, i.e. the system $S_1 \oplus S_2 \oplus S_3$, to be in a state something like

$$|\Psi\rangle = \sum_{x \in \mathbf{Sp}(\hat{X})} \lambda_x \sum_{t \in \mathbf{Sp}(\hat{T})} \mu_t \left[\theta(t<0)|x,\, 0\rangle + \theta(0 \le t \le T) e^{i\hat{H}t}|x,\, 0\rangle + \theta(t>T)|x,\, x\rangle \right] |t\rangle \tag{8}$$

where θ is the function that takes the value 1 when its argument is a true proposition, and 0 otherwise. $|\Psi\rangle$ is a fixed state with no time-dependence in the usual sense. Nevertheless if we choose to refer to the eigenvalues of $\hat{T}$ as "times", the (Everett) interpretation of $|\Psi\rangle$ that we read off from successive terms in its expansion (8) does describe motion:

At times before 0, the apparatus observable $\hat{A}$ has the receptive value 0 and $\hat{X}$ is multi-valued. Between times 0 and T, $\hat{A}$ becomes multi-valued in a way that is correlated with $\hat{X}$. After time T, in each branch $\hat{X}$ has its original value and $\hat{A}$ has that same value.

The complex amplitudes μ_t are arbitrary except that they satisfy

$$\sum_{t \in \mathbf{Sp}(\hat{T})} |\mu_t|^2 = 1 \tag{9}$$

in order to normalize $|\Psi\rangle$, and that presumably none of them vanishes. I have argued elsewhere that there is no physical reason why a state such as $|\Psi\rangle$ must be normalizable with respect to the sum over times t because the weight $|\mu_t|^2$ of an Everett branch corresponding to a particular time is not the probability of anything. However if we were to allow non-normalizable states we should have to go to the trouble of amending the Hilbert space formalism to give meaning to representations such as (8) when the sum in (9) is divergent. That is not worth doing for our present purposes because we shall not encounter any problem in normalizing $|\Psi\rangle$.

To say that $\mathbf{Sp}(\hat{T})$ should physically approximate the real line is to say that there should be many eigenvalues t of $\hat{T}$ in any interval over which quantities of interest vary significantly as functions of t. Therefore the sums over t in (8) and (9) should be replaceable by integrals.

We shall take

$$\mu_t = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha t^2} \tag{10}$$

where α is very small and positive so that μ_t varies very slowly over the interval of interest ($0 \le t \le T$) but nevertheless falls rapidly to zero as $t \to \pm\infty$. Any function with those properties would serve equally well in what follows.

Let $\hat{P}$ be the system-S_3 projection operator

$$\hat{P} = \sum_{\substack{t \in \mathbf{Sp}(\hat{T}) \\ 0 \le t \le T}} |t\rangle\langle t| \approx \int_0^T |t\rangle\langle t| \, dt \tag{11}$$

for the time to lie between 0 and T, i.e. for the period of the measurement. Consider the Hamiltonian

$$\hat{\mathbf{H}} = \hat{H} \otimes \hat{P} + \hat{1} \otimes \hat{h} \tag{12}$$

for the universe $(S_1 \oplus S_2) \oplus S_3$. Like $|\Psi\rangle$, this has no dependence on any time parameter — yet if we use the term "evolve" to mean "change to successive eigenvalues of $\hat{T}$", we can say the following about the dynamics of a universe governed by the Hamiltonian $\hat{\mathbf{H}}$:

System S_3 evolves independently under the Hamiltonian $\hat{h}$. Systems S_1 and S_2 evolve under the Hamiltonian $\hat{H}$ in all branches in which $\hat{T}$ has values between 0 and T, and do not evolve at all otherwise.

Now it is easily seen that under the approximations stated above (i.e. sums over t are replaced by integrals and α is very small), $|\Psi\rangle$ is a solution of the Schrödinger equation for a system with Hamiltonian $\hat{\mathbf{H}}$ — specifically it is an eigenstate of $\hat{\mathbf{H}}$ with eigenvalue zero. From (3), (8), (11) and (12),

$$\hat{\mathbf{H}}|\Psi\rangle = -i \sum_{x \in \mathrm{Sp}(\hat{X})} \lambda_x \int_{-\infty}^{\infty} \mu_t \frac{d}{dt}\left\{ \left[\theta(t<0)|x, 0\rangle + \theta(0 \le t \le T) e^{i\hat{H}t}|x, 0\rangle + \theta(t>T)|x, x\rangle\right] |t\rangle \right\} dt. \quad (13)$$

Because of the properties of μ_t as $t \to \pm\infty$, the boundary term on integration by parts vanishes, so

$$\hat{\mathbf{H}}|\Psi\rangle = i \sum_{x \in \mathrm{Sp}(\hat{X})} \lambda_x \int_{-\infty}^{\infty} \frac{d\mu}{dt}\left[\theta(t<0)|x, 0\rangle + \theta(0 \le t \le T) e^{i\hat{H}t}|x, 0\rangle + \theta(t>T)|x, x\rangle\right] |t\rangle dt. \quad (14)$$

Now, taking the kets $\{\,|t\rangle\}$ in (14) to be orthonormal (remember that the integral over t is really a sum over the values for which they *are* orthonormal) and using (7) and the orthonormality of the eigenstates of $\hat{X}$, we have

$$\left\|\hat{\mathbf{H}}|\Psi\rangle\right\|^2 = \int_{-\infty}^{\infty} \left|\frac{d\mu}{dt}\right|^2 dt = \alpha \to 0, \quad (15)$$

as stated.

There is one difference between this model and that of Page and Wooters. In their model the clock and the other subsystem were strictly non-interacting. In this model the "clock", which is S_3, the "rest of the universe", does interact with the other two systems (or rather, it acts on them and they do not react back) and plays a realistic role in the measurement process.

Twistor regularisation of ultra-violet divergences

In **TN 29**, and in my **TMP** review, I suggested that the introduction of *inhomogeneous boundaries at infinity* into twistor diagrams should have the potential to eliminate ultra-violet divergences, while retaining a manifestly finite integral formalism. It's now possible to show this for a special case of such a divergence, namely the Feynman diagram

i.e. $$\int d^4x\, d^4y\; \phi_1(x)\phi_2(x)\left(\Delta_F(x-y)\right)^2 \phi_3(y)\phi_4(y) \tag{1}$$

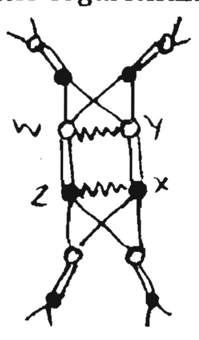
in massless ϕ^4 theory. To do this I've gone back to the argument sketched out in TN 25 . This argument was basically on the right lines, but what I didn't see then was the essential role of *conformal symmetry breaking* in higher order Feynman diagram calculations - and this is the key factor.

In fact I should have noted that it's obvious that some such symmetry-breaking *must* come in. The regularisation of this divergent integral, as achieved by conventional QFT methods, is of the form

$$\log(p^2/\mu^2)$$

where p is the total ingoing (and outgoing) momentum and μ is some arbitrary mass. This doesn't just break conformal invariance: it's not even scale invariant. Note that although one may not think of ϕ^4 theory as geunine physics, the integral being studied here is essentially the same as

in QED, and that the logarithmic factor in *that* context corresponds to the [zero-mass limit of the] Lamb shift - very well corroborated by experiment. So we should consider the logarithm as a genuine physical feature, making it imperative that some scale-breaking mechanism must be introduced. In fact it's not hard to write down a twistor diagram which does this and yields agreement with the logarithmic answer, namely



where the boundaries are on $\bar{Z}X = \mu$, $WY = \mu$, i.e. they are inhomogeneous boundaries at infinity, capable of breaking the scale invariance.

This is very encouraging, as it agrees with the general "skeleton" pattern postulated for the twistor version of Feynman diagrams. But can this diagram be *derived* - not just written down ***ad hoc*** using knowledge of the conventional regularised answer?

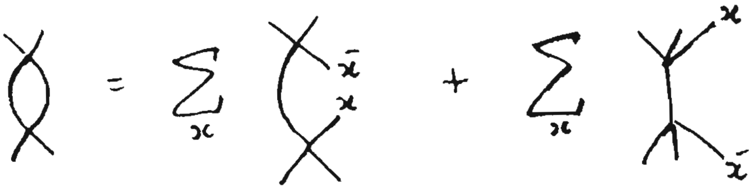
To analyse the problem, first note that in momentum space the Feynman integral appears as the (divergent) integral

$$\int \frac{d^4k}{k^2(p-k)^2}$$

where the integration is to be done according to the Feynman prescription. By elementary complex analysis, this prescription means that at least formally it is the same as

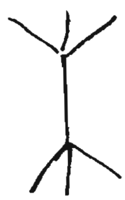
$$\int \frac{d^4k\,\delta^+(k^2)}{(p-k)^2} + \int \frac{d^4k\,\delta^+((p-k)^2)}{k^2}$$

Here the δ^+ functions are just on shell propagators, which can be thought of as sums over a complete set of free states; thus we have

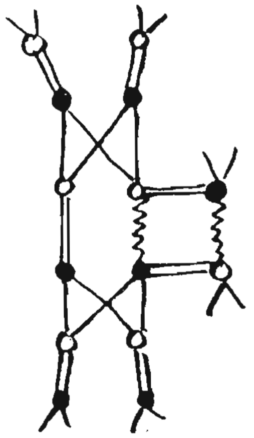


These sums are divergent, but the tree diagrams themselves are supposed to be finite, and the next thing is to study these tree diagrams in detail.

Much of this analysis has already been done in TN 25, and so I shall here simply assert that using the information described there, the ϕ^4 diagram



is a finite, conformally invariant functional of the fields and can be represented exactly by the twistor diagram:



(2)

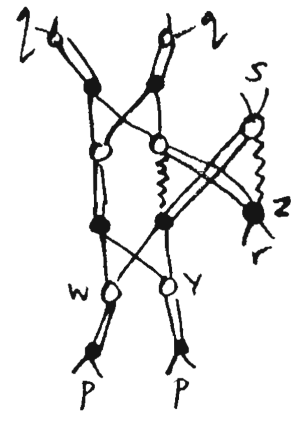
It's quite another story with the other channel. Let's be specific and use particular elementary states. Without loss of generality we can consider

$$\int \frac{d^4x\, d^4y\, \Delta_F(x-y)}{((x-p)^2)^2\,(x-s)^2\,((y-q)^2)^2\,(y-r)^2} \qquad (3)$$

where p, r are in the past tube and q, s in the future tube. This must yield a function F(p ,q ,r ,s) satisfying

$$\left(\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p} + 2\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial s}\right)F(p,q,r,s) = \int \frac{d^4x}{((x-p)^2)^2\,(x-s)^2\,((x-q))^2\,(x-r)^2} \qquad (4)$$

The first difficulty is that the Feynman integral (3) is divergent, a problem swept under the carpet in the conventional approach where $1/k^2$ is called "finite" although it's singular at $k^2 = 0$. This means that we are driven first to find ***a regularisation for this tree diagram*** - a procedure quite unlike the conventional approach. In doing this we can be guided by the regularisation of the Møller scattering divergence. This naturally suggests the possibility that the divergence encountered here is regularised by the inhomogeneous twistor integral:



(5)

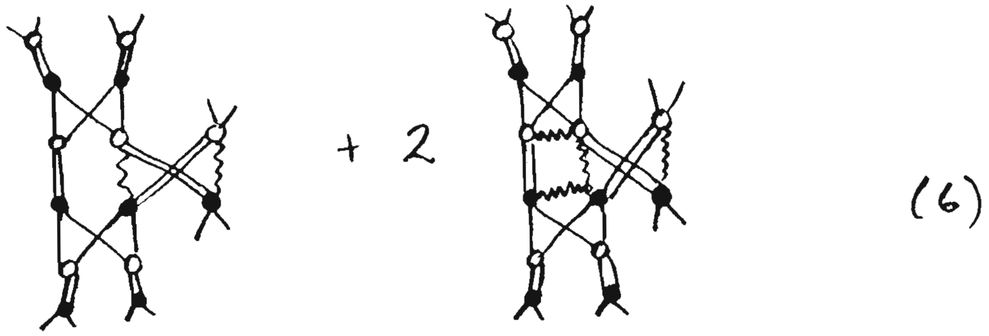
Calculation shows however that this doesn't satisfy the differential equation (2). In fact the two sides of the equation fail to match by [a multiple of]

$$\{((p-q)^2)^2\,(q-r)^2\,(p-s)^2\}^{-1}$$

It follows that if we put

$$F(p,q,r,s) = \{(5)\} + 2\,\frac{\log((p-q)^2)}{(p-q)^2(q-r)^2(p-s)^2}$$

then this new F satisfies the essential equation (4). Note that a scale breaking element has entered now. It now turns out that this revised candidate for the regularised tree amplitude can be put in the form



where the extra term contains inhomogeneous boundaries at infinity to do the scale breaking. This looks very promising! It appears that we can now sum over the states as required, i.e. replace

The conformally invariant expressions cancel leaving just the contribution from the scale-breaking part. Unfortunately this leads to exactly TWICE the right answer (and so twice the right Lamb shift.) What's gone wrong? The trouble is that we haven't shown that (6) is a genuine regularisation of the divergent Feynman integral in (3); there could be other regularisations which differ by solutions of the homogeneous equation

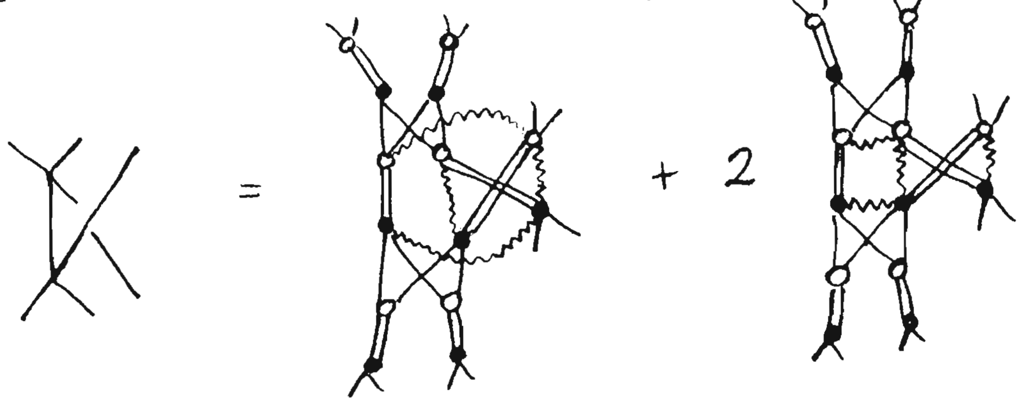
$$\left(\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p} + 2\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial s}\right)F(p,q,r,s) = 0$$

Indeed (6) is NOT a genuine regularisation. This is demonstrated by the fact that the interior of the diagram (5) doesn't satisfy the spin-0 eigenstate condition, i.e. that it's an eigenstate with eigenvalue 0 of

$$(Y.Z)\left(\frac{\partial}{\partial Y}\cdot\frac{\partial}{\partial Z}\right)$$

whilst the scale-breaking diagram added on in (6) does satisfy it. This means that the total functional of fields represented by (6) doesn't project out the spin-0 part of the fields meeting at a vertex - as it must to be a genuine regularisation of (3).

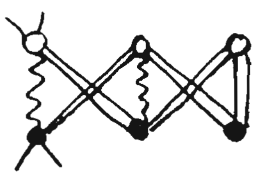
To cut a long story short, there DOES exist another completely finite functional of the fields which satisfies both (4) and the relevant spin eigenstate conditions. It can be represented by:



(7)

Note that scale-breaking inhomogeneous boundaries at infinity come into the integral thus introduced. The resulting total functional of fields is still not uniquely fixed by these conditions, so some further characterisation of satisfactory regularisation is still required. But I will assume that this is in fact the right answer for the tree diagrams.

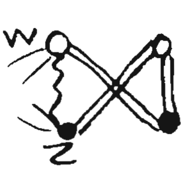
Now we can sum over states. The essential idea here is that these divergent sums are also regularised by twistor diagram inhomogeneity - but this time we only need a version of the "Møller" mechanism. For instance we can evaluate



as $\log\left(\frac{kK}{kk}\right)$

$= 0$

The basic reason for the finite answer is that in integrating out the states,



becomes $\oint_{W.Z>k} \frac{DWZ}{(W.Z)^2}\cdots$

so that the 'k' saves the pole from meeting the boundary. This mechanism can be applied consistently and this time we get the *right* answer for the loop integral, as there is now a partial cancellation between the scale-breaking terms.

I hope this analysis can be generalised to encompass all divergences (including vacuum diagrams) systematically, but much more work is needed yet.

Andrew Hodges

Preferred parameters on curves in conformal manifolds

Toby Bailey Michael Eastwood

October 2, 1990

What follows are some observations made while considering whether one can construct analogues in conformal differential geometry of the 'pinched curvature, injectivity radii and minimising geodesics' ideas that lead (for example) to the Sphere Theorem in Riemannian differential geometry. These considerations are at a very early stage, but have led indirectly to an interesting (and as far as we know original) result about the distribution of curvature on a closed curve in $\mathbb{R}^n$.

Let γ be a *curve* (i.e. a smoothly immersed 1-dimensional submanifold) in an n-dimensional conformal Riemannian manifold. Choose a metric in which to work and parametrise γ by an arc-length parameter t and let U^a be the unit tangent vector. The *curvature* is given by $\kappa = \sqrt{A^b A_b}$, where $A^b = U^a \nabla_a U^b$.

As observed by Cartan (for an account in this language see the authors' paper in Proc. AMS **108** (1990), 215–221), such a curve has a natural *projective structure*—i.e. a family of preferred parameters related by fractional linear transformations under $SL(2, \mathbb{R})$. The function s on γ is a preferred parameter if it obeys the ('inhomogeneous Schwarzian') equation

$$(s')^{-1}s''' - \tfrac{3}{2}(s')^{-2}(s'')^2 = \tfrac{1}{2}\kappa^2 + P$$

where 'dash' denotes differentiation with respect to t and $P = P_{ab}U^aU^b$ where P_{ab} is the usual trace-modified multiple of the Ricci tensor. The simplified form of the equation when compared with the above reference is due to our use of an arc-length parameter.

In order to study the behaviour of these parameters we substitute $\xi = s(s')^{-1/2}$ since a brief calculation shows that these *ξ-parameters* obey the linear equation

$$\xi'' + (\tfrac{1}{4}\kappa^2 + \tfrac{1}{2}P)\xi = 0.$$

We define the *index* of a curve from A to B to be the number of zeroes (excluding the initial one) that the ξ-parameter with $\xi(A) = 0, \xi'(A) = 1$ has *before* reaching B. The location of, (and hence the number of) such zeroes is conformally invariant. As a first step towards investigating the properties of this index we have considered its behaviour on closed curves in $\mathbb{R}^n$. If γ is a (geometric) circle, it is easy to check that the first zero of any ξ-parameter occurs exactly at the starting point after one full traverse of the curve. As we see below, this characterises the circles among all closed curves.

We begin with a result (perhaps of interest in its own right) about the distribution of curvature on a closed curve.

Proposition *Let γ be a closed curve in $\mathbb{R}^n$ of length l, parametrised by arc-length t. Let κ be the curvature of γ. Then*

$$\int_0^l \kappa^2 \sin^2\left(\frac{\pi t}{l}\right) dt \geq \frac{2\pi^2}{l}$$

with equality if and only if γ is a circle.

The proposition is proved by expanding the coordinates as functions of t in Fourier series and performing some essentially trivial manipulations.

Proposition *Let γ be a closed 1-dimensional submanifold of $\mathbb{R}^n$ which is not a geometrical circle. Then any closed curve which consists of traversing γ once from some chosen starting point has index at least 1.*

Proof Let γ be of length l and consider the eigenvalue problem

$$\left(-\frac{d^2}{dt^2} - \frac{1}{4}\kappa^2\right)\xi = \lambda\xi, \qquad \xi(0) = \xi(l) = 0$$

on the interval $[0, l]$. Then the ξ-parameter with $\xi(0) = 0, \xi'(0) = 1$ will have a zero before $t = l$ if and only if zero is greater than the least eigenvalue of this problem. Since the operator on the left-hand side is bounded below we know that the least eigenvalue is always less than

$$\int_0^l \phi(t)\left(-\frac{d^2}{dt^2} - \frac{1}{4}\kappa^2\right)\phi(t)dt$$

for any function the integral of whose square over $[0, l]$ is unity.

Taking $\phi(t) = \sqrt{2/l}\sin^2(\pi t/l)$ we see that a sufficient condition is that

$$\int_0^l \kappa^2 \sin^2(\pi t/l)dt \geq \frac{2\pi^2}{l}.$$

The result then follows from Proposition 1. □

It is unclear at this stage whether these ideas (together perhaps with a study of the 'exponential map' for conformal circles) will lead to any interesting results on conformal manifolds. It would be interesting to know (for example) whether conformal circles are in any sense index-minimising curves. As a starting point however one can (easily) prove results such as:

Proposition *If M is a compact Riemannian conformal manifold such that there is a metric in the conformal class with pinched sectional curvatures*

$$\frac{2(n-1)}{4(n-2)k^2 + n} \leq K \leq 1$$

and $k \geq 1$ is an integer, then any two points can be joined by a curve of index less than k.

The authors thank John Baez for assistance with the proof of the second proposition.

Toby Bailey and Michael Eastwood

Self-dual manifolds need not be locally conformal to Einstein

There are global topological obstructions to the existence of Einstein metrics in four dimensions (which imply, for example, that $\mathbb{CP}_2 \# \mathbb{CP}_2 \# \mathbb{CP}_2 \# \mathbb{CP}_2$ does not admit such a metric). On the other hand, $\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2$ always admits a self-dual metric (as shown abstractly by Donaldson & Friedman [2] and explicitly by LeBrun [5]). The purpose of this note is to observe that there are also local obstructions to the existence of an Einstein metric within a given self-dual conformal class. We shall discuss the obstruction

$$K_{abc} = C^{efgh}C_{efgh}\nabla^d C_{abcd} - 4C^{efgh}C_{abch}\nabla^d C_{efgd}$$

of Kozameh, Newman, & Tod [4] where C_{abcd} is the Weyl tensor. This tensor is easily shown to be conformally invariant and the contracted Bianchi identity shows that it vanishes in the case of an Einstein metric. If C_{abcd} is self-dual, then it may be written as $K_{abc} = -4K_{A'B'C'C}\,\varepsilon_{AB}$ where

$$K_{A'B'C'C} = \tilde{\Psi}^{E'F'G'H'}\tilde{\Psi}_{E'F'G'H'}\nabla_C^{D'}\tilde{\Psi}_{A'B'C'D'} - 2\tilde{\Psi}^{E'F'G'H'}\tilde{\Psi}_{A'B'C'H'}\nabla_C^{D'}\tilde{\Psi}_{E'F'G'D'}.$$

In [1], Baston & Mason identified two tensors

$$E_{abc} = \Psi_{ABCD}\nabla^{DD'}\tilde{\Psi}_{A'B'C'D'} - \tilde{\Psi}_{A'B'C'D'}\nabla^{DD'}\Psi_{ABCD}$$
$$B_{ab} = (\nabla_{A'}^{C}\nabla_{B'}^{D} + \Phi_{A'B'}^{CD})\Psi_{ABCD}$$

whose vanishing in the case of algebraically general Weyl curvature is necessary and sufficient for the existence of an Einstein scale. These tensors evidently vanish for a self-dual metric. It is, therefore, enlightening to notice that

$$2\Psi_C{}^{EFG}(\delta_{A'}^{E'}\delta_{B'}^{F'}\delta_{C'}^{G'}\tilde{\Psi}^2 - 2\tilde{\Psi}^{E'F'G'H'}\tilde{\Psi}_{A'B'C'H'})E_{efg} = \Psi^2 K_{A'B'C'C}$$

(where $X^2 = X^{ABCD}X_{ABCD}$). Thus, if E_{abc} vanishes and the Weyl curvature is algebraically general (whence, in particular, Ψ^2 is nowhere vanishing), then K_{abc} also vanishes.

We claim, however, that there are self-dual metrics for which K_{abc} is non-zero and hence a genuine obstruction to the existence of an Einstein scale. In fact, the metric

$$(dw + 2xy\,dy - x\,dz)^2 + 2y^2z(dx^2 + dy^2) + y^2dz^2$$

is self-dual but K_{abc} is nowhere vanishing — for example,

$$K \lrcorner \left(\frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x}\right) = \frac{9}{8y^7z^4} \quad \text{and} \quad K \lrcorner \left(\frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}\right) = \frac{9x(y^2-2z)}{4y^8z^5}.$$

The explicit computation of K_{abc} is far from easy. These results were obtained via a computer program (written in "maple" and available upon request: it computes K_{abc} along with other differential geometric fauna for any explicit metric and can also check whether the metric is self-dual).

Of course, the metric above was constructed to be self-dual — notice that $\partial/\partial w$ is a Killing vector so the metric arises from an Einstein-Weyl space together with a generalized monopole as described by Jones & Tod [3]. In fact, the Einstein-Weyl space in question is hyperbolic 3-space. This is also the basis of LeBrun's explicit metrics [5] on $\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2$. The generalized monopoles that he employs, however, are derived from Green's functions for the hyperbolic Laplacian. Whilst extremely natural, they are computationally more difficult and we have not yet succeeded in completing the calculation of K_{abc} in this case. We suspect that it will turn out to be non-zero.

The tensor K_{abc} is evidently an obstruction to rescaling the metric so that $\hat{\nabla}^d \hat{C}_{abcd} = 0$, a so-called "C metric". Indeed, its derivation in [4] is from

$$\nabla^d C_{abcd} + \Upsilon^d C_{abcd} = 0, \qquad *$$

noting that if $C^2 \equiv C^{abcd}C_{abcd} \neq 0$, then $\Upsilon^h = -4\,C^{efgh}\nabla^d C_{efgd}/C^2$. Substituting back into $*$ gives K_{abc}/C^2. In the algebraically general case Υ^h is automatically closed [4]. We presume this is an extra condition in s-d case.

We thank Claude LeBrun for many helpful communications.

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Toby Bailey and Michael Eastwood

Families of invariants

There are many well known examples of linear conformally invariant differential operators: For a flat four dimensional conformal geometry the Laplacian $\Delta := \nabla^a \nabla_a$ is invariant when acting on densities of weight -1. If f is a density of weight 1 then

$$\nabla_{(a} \nabla_{b)_0} f \tag{1}$$

(where $()_0$ denotes the trace-free symmetric part) is invariant.

One can also construct non-linear invariants. For example, in four dimensions, if again f has weight 1 then

$$f \Delta f - 2 \nabla^a f \nabla_a f \tag{2}$$

is invariant. It is interesting to note that this latter invariant is closely related to the Laplacian invariant via the identity

$$-f^3 \Delta f^{-1} = f \Delta f - 2 \nabla^a f \nabla_a f.$$

Indeed we can use this to deduce the invariance of (2) from the invariance of the Laplacian since f^{-1} has weight -1. More generally if f has weight w then $f^{2+1w} \Delta f^{-\frac{1}{w}}$ is clearly invariant. Expanding this out we obtain a family of invariants parametrised by weight w:

$$w f \Delta f - (w+1) \nabla^a f \nabla_a f.$$

The Laplacian, or at least $f \Delta f$, is seen to be just a special case of this family. By similar reasoning the invariant (1) is found to be a special case (if we ignore overall left multiplication by f) in the family

$$w f \nabla_{(a} \nabla_{b)_0} f - (w-1) \nabla_{(a} f \nabla_{b)_0} f.$$

Indeed given any invariant on (non-zero weight) densities, which is polynomial in jets of the density, one can use this technique to generate the family to which it belongs. In other words provided we avoid invariants of the functions[1] then all invariants on densities of any given weight can be

[1] In fact the reader will observe that putting $w = 0$ in the above formulae does yield an invariant. However at this stage it is not known how many invariants of functions do not arise in this fashion. $\nabla_a f^a$ is one example.

obtained in this manner from the set of all invariants on any other given weight.

For invariants of densities this procedure and these results also work in other dimensions and for other structures (for example projective geometries) and also for the corresponding curved cases. On the other hand it is difficult to imagine that *this* scheme for producing the families of invariants can be generalised to deal with quantities other than densities (i.e. weighted tensors and spinors). However, at least in the flat case, these families can be generated in other ways [1] that work equally well for tensors. It is likely that similar results hold for this more general case – i.e. that all invariants occur in families. If this is true then the problem of producing a complete theory of invariants is considerably reduced.

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Quaternionic complexes.

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Abstract

Each regular or semi-regular integral affine orbit of the Weyl group of $\mathrm{gl}(2n+2,\mathbb{C})$ invariantly determines a locally exact differential complex on a $4n$ dimensional quaternionic manifold.

Duke Math J.:

Almost Hermitian Symmetric Manifolds I Local Twistor Theory

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Abstract

Conformal and projective structures are examples of structures on a manifold which are modelled on the structure groups of Hermitian symmetric spaces. We show that each such structure has associated a distinguished vector bundle (or *local twistor bundle*) equipped with a connection (*local twistor transport*). For projective and conformal manifolds, this is Cartan's connection. The curvature of the connection provides an tensor invariant which vanishes if and only if the manifold is locally isomorphic to a Hermitian symmetric space.

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Duke Math J. to appear:

Almost Hermitian Symmetric Manifolds II Differential Invariants

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Abstract

We use local twistor connections and Lie algebra cohomology to construct linear differential operators depending invariantly on almost Hermitian symmetric structures. All *standard* homogeneous (flat) operators admit curved analogues whilst most *nonstandard* ones appear obstructed; these obstructions yield further invariants of the AHS structure. The methods of the paper, applied in the flat case to irreducible quotients of Verma modules, construct nonstandard homomorphisms of Verma modules given (relative) Kazhdan–Lustzig polynomials.

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Zuckerman functors, the Penrose transform and homomorphisms of Verma modules.

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A multiplicity one theorem for Zuckerman's functors

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Abstract

Let $\mathbf{g}$ be a complex semisimple Lie algebra and $\mathbf{p}, \mathbf{r}$ standard parabolic subalgebras with $\mathbf{k}$ a reductive Levi factor of $\mathbf{p}$. We prove, via the *Penrose transform* and results of Enright–Shelton on semi-regular categories, that irreducibles occur with multiplicity at most one in the cohomologically induced modules $\oplus_i \Gamma^i_{\mathbf{k}} N$. Here N is a (dual) Verma module induced from $\mathbf{r}$, and $\mathbf{p}, \mathbf{r}$ are of a restricted class which includes all Hermitian symmetric cases. This, with Vogan's U_α calculus, gives an effective method for computing the irreducible subquotients of the $\Gamma^i_{\mathbf{k}} N$.

The methods used generalize to a geometric setting, in which $\Gamma^i_{\mathbf{k}} N$ is replaced by holomorphic sheaf cohomology over an open subvariety of G/R and may be viewed as an extension of the Bott–Borel–Weil theorem to such domains.

A multiplicity one theorem for the Penrose transform

In the last TN, I outlined a method for computing the cohomology of certain sheaves on homogeneous twistor spaces Z for general complex semisimple Lie groups. I used the fact that (a) the sheaves in question are related to each other by certain explicit short exact sequences and (b) there is a basic sheaf (obtained by helicity raising from the canonical bundle) whose cohomology is simple to compute[1]. The only tricky point is to understand the maps in the resulting long exact sequences on cohomology. It turns out that there is an elegant inductive procedure for doing this in a good number of cases, including all of physical interest. The result may be viewed as an elegant extension of the Bott-Borel-Weil theorem to *open* homogeneous spaces. It also removes spectral sequences in the Penrose transform from recipes to their proofs.

Suppose we have a short sequence

$$0 \to A \to B \to C \to 0$$

obtained by helicity lowering and raising as in the last TN. Suppose we know $H^i(Z, A)$ and so $H^i(Z, B)$. To obtained $H^i(Z, C)$ from the long exact sequence

$$\cdots \to H^i(Z, A) \xrightarrow{\alpha} H^i(Z, B) \to H^i(Z, C) \to \cdots$$

we need to know α. These cohomology groups bear extra structure because $Z \subset G/R$ is a subset of a complex homogeneous space and A, B, C are homogeneous (see [2] for terminology etc.). They are all modules over the Lie algebra $\mathfrak{g}$ of G—in the usual four dimensional twistor picture, $G = \mathrm{SL}(4, \mathbf{C})$ covers the complex conformal group. We can therefore try to decompose $H^i(Z, A)$ etc. into $\mathfrak{g}$ irreducibles. If $\mathfrak{g}$ were the real Lie algebra of a compact Lie group the result would be a direct sum of finite dimensional vector spaces, each an irreducible representation of $\mathfrak{g}$. Since $\mathfrak{g}$ is complex, however, the best we can do is to write $H^i(Z, A)$ as a tower of submodules whose successive quotients are direct sums of (not necessarily finite dimensional) irreducible representations. We get a picture of $H^i(Z, A)$ as a building with each storey a home for a direct sum of irreducibles. Thus we might have

$$H^i(Z, A) \quad = \quad \begin{array}{|c|} \hline L_4 \oplus L_5 \\ \hline L_2 \oplus L_3 \\ \hline L_1 \\ \hline \end{array}$$

with L_1 a submodule. By requiring that A is irreducible, we can ensure that the L_i come from a known finite list of possibilities. The scheme is to figure out what α does to each L_i. The ideal answer is the following

[1] Similar ideas were suggested by Lionel Mason in an earlier TN.

Proposition *If L_i occurs in $H^i(Z, A)$ and in $H^i(Z, B)$ then α is an isomorphism between them.*

A much stronger result is in fact true. Recall that Z corresponds to a Stein open subset $X \subset G/P$ of a second homogeneous space. Suppose (G, P) and (G, Q) are both Hermitian symmetric pairs (this includes all cases of physical interest—see [2, chap 10]) and A is now any irreducible homogeneous sheaf on Z. Then

Theorem *An irreducible* $\mathfrak{g}$ *module L occurs in at most one degree in any $H^i(Z, A)$ and then only once. There is an algorithm for determining when it occurs.*

This certainly implies the proposition, for if α is zero on an L occuring in both $H^i(Z, A)$ and $H^i(Z, B)$ then L occurs in $H^{i-1}(Z, C)$ and $H^i(Z, C)$. The algorithm follows from the long sequences.

The idea behind the proof is quite simple. There are certain special homogeneous bundles D called *singular* on G/R, distinguished by $H^i(G/R, D) = 0$ for all i. $\mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1)$ are all examples for ordinary twistor space in $\mathbf{CP}^3$. Indeed, any non singular bundle A can be sent to such a bundle by helicity lowering. The short exact sequences above are obtained by first doing this lowering and then raising back to the original helicity. Taking cohomology commutes with this helicity lowering. If L had multiplicity more than one in $H^i(Z, A)$, we could arrange to helicity lower A to a singular bundle D so that the helicity lowered L is not zero and still has multiplicity more than one in $H^i(Z, D)$. This means that proving the theorem reduces to proving it for such singular bundles.

A great deal is known about singular bundles in the Hermitian symmetric setting. It turns out [3] that they behave very much like *non singular* bundles for groups G', P', R' of the same kind as G, P, R *but smaller dimension!* The proof of the theorem now reduces checking that this behaviour extends to the Penrose transform, for that provides an inductive step on the rank (dimension) of G. Some rather beautiful subtleties emerge whilst doing this to explain how cohomology degrees are related between the pictures for G' and G. Details to come in [1].

Thanks to Brad Shelton for explaining the philosophy behind [3] and Jim Isenberg for hospitality in Eugene, OR. I won't forget the mountain bike ride guys!

Rob Baston

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A SPINOR FORMULATION FOR HARMONIC MORPHISMS

by

Paul Baird(*) and John C. Wood

0. Introduction

The aim of this paper is to draw attention to the simple description of harmonic morphisms in terms of spinors, and to interpret the equations in terms of holomorphicity properties of sections of twistor bundles.

Harmonic morphisms have been studied by mathematicians for some time. They are defined as mappings between Riemannian manifolds which pull back germs of harmonic functions to germs of harmonic functions. Equivalently they are the harmonic mappings which are horizontally conformal (see [B1, B3, BW1] for details and further references). Thus if $\phi : M \to \mathbf{C}$ is a mapping from a Riemannian manifold M ($\dim M \geq 2$) with values in the complex plane $\mathbf{C}$, then ϕ is a harmonic morphism if and only if

$$g^{ab} \frac{\partial \phi}{\partial x^a} \frac{\partial \phi}{\partial x^b} = 0 \tag{0.1}$$

$$\Delta\phi = g^{ab}\left(\frac{\partial^2 \phi}{\partial x^a \partial x^b} - \Gamma^c_{ab} \frac{\partial \phi}{\partial x^c} \right) = 0 \quad , \tag{0.2}$$

where $g = g^{ab}$ is the metric on M and the Γ's are the corresponding Christoffel symbols. The first equation expressing horizontal conformality, the second harmonicity. In this note we concentrate on the case when $M \subset \mathbf{R}^4$ is an open subset of Euclidean 4-space. At the end we indicate the Euclidean $\mathbf{R}^3$ case and the Minkowski M^4 case.

To a harmonic morphism $\phi : M \to \mathbf{C}$, $M \subset \mathbf{R}^4$, we associate a pair of spinor fields $(\xi^A, \eta^{A'})$ defined on M. These satisfy the spinor equations

$$\begin{cases} \nabla_{AA'} \xi^A \eta^{B'} = 0 \\ \nabla_{AB'} \xi^C \eta^{B'} = 0 \, . \end{cases} \tag{1.7}$$

We interpret the projectivised fields $[\xi^A]$, $[\eta^{A'}]$ in terms of Gauss sections. The pair $([\xi^A], [\eta^{A'}])$ then determines a section of the well known twistor bundle $Z^+ \times Z^-$ over M (see [ES]). This ties in with the description of the second author [W] for submersive harmonic morphisms from a Riemannian 4-manifold to a surface. The equations (1.7) are then equivalent to holomorphicity equations for that section. It is worth pointing out that the spinor formulation in this note does not require the restriction that ϕ be submersive. This was necessary in [W] to guarantee a decomposition of the tangent space into well-defined vertical and horizontal spaces at each point. Thus we generalize here to arbitrary harmonic morphisms.

As an additional comment we note that a submersive harmonic morphism from a Riemannian m-dimensional manifold M to a surface is locally equivalent to an (m-2)-dimensional conformal foliation of M by minimal submanifolds. In the case when $m = 3$, we can remove the

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restriction 'submersive' and the foliation is by geodesics. Such conformal foliations are the Riemannian analogue of the well known shear-free null geodesic congruences, much studied by relativists in connection with zero rest mass fields (see [BW2, 3] for details).

In [BW1,2] all harmonic morphisms from a three-dimensional simply connected space form to a surface were determined. In these cases the harmonic morphisms are determined by pairs of holomorphic functions, see also Tod [T1, 2].

Throughout we use spinors as described in the Appendix of [PR, vol 2]. This enables us to consider spinors defined on a vector space with metric of arbitrary signature.

1. Harmonic morphisms from $\mathbf{R}^4$ in terms of spinors

We consider $\mathbf{R}^4$ with its standard Euclidean metric. Vectors x^a may be expressed in terms of spinors by the correspondence:

$$(x^0, x^1, x^2, x^3) \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} i\,x^0 + x^1 & x^2 + i\,x^3 \\ x^2 - i\,x^3 & ix^0 - x^1 \end{pmatrix} = x^{AA'}.$$

Writing $\partial_a = \partial/\partial x^a$, the spinor covariant derivatives $\nabla_{AA'}$ are given by

$$\nabla_{00'} = \frac{1}{\sqrt{2}}(-i\,\partial_0 + \partial_1)$$
$$\nabla_{01'} = \frac{1}{\sqrt{2}}(\partial_2 - i\,\partial_3)$$
$$\nabla_{10'} = \frac{1}{\sqrt{2}}(\partial_2 + i\,\partial_3)$$
$$\nabla_{11'} = \frac{1}{\sqrt{2}}(-\,i\,\partial_0 - \partial_1)\ .$$

Now let $M \subset \mathbf{R}^4$ be an open subset, and recall that $\phi : M \to \mathbf{C}$ is *horizontally conformal* if and only if

(1.1) $$\sum_a \left(\frac{\partial\phi}{\partial x^a}\right)^2 = 0\ ,$$

and ϕ is *harmonic* if and only if

(1.2) $$\sum_a \frac{\partial^2\phi}{(\partial x^a)^2} = 0.$$

So ϕ is a harmonic morphism if and only if (1.1) and (1.2) are satisfied. Let $\phi : M \to \mathbf{C}$ be a smooth mapping. From equation (1.1) we immediately deduce that ϕ is horizontally conformal if and only if

(1.3) $$\nabla_{AA'}\phi = \xi_A\eta_{A'}\ ,$$

for some spinor fields $\xi_A, \eta_{A'}$ defined on M.

Remarks 1) We always have the freedom $(\xi_A, \eta_{A'}) \to (\lambda\xi_A, (1/\lambda)\eta_{A'})$, $\lambda \in \mathbf{C}$.

2) At a critical point of ϕ, $\nabla_{AA'}\phi = 0$ and one of $\xi_A, \eta_{A'}$ is zero.

3) Equation (1.1) is the condition that the gradient be a complex null vector field. That is $\nabla\phi.\nabla\phi = 0$.

Since (1.2) is equivalent to $\nabla^{AA'}\nabla_{AA'}\phi = 0$, we have that, if ϕ is horizontally conformal, so that $\nabla_{AA'}\phi = \xi_A\eta_{A'}$ for spinors $\xi_A, \eta_{A'}$, then ϕ is harmonic if and only if

(1.4) $$\nabla^{AA'}\xi_A\eta_{A'} = 0 .$$

Conversely, given a pair of spinor fields $\xi_A, \eta_{A'}$ on M, we would like conditions which ensure they determine a harmonic morphism. Now the product $\xi_A\eta_{A'}$ determines a null vector field v_a. We require $\nabla_{[a}v_{b]}$ to be zero. This is computed to be equivalent to the six spinor equations:

(1.5) $$\begin{cases} \text{(i)} & \nabla_{11'}\xi^1\eta^{1'} - \nabla_{00'}\xi^0\eta^{0'} = 0 \\ \text{(ii)} & \nabla_{01'}\xi^0\eta^{1'} - \nabla_{10'}\xi^1\eta^{0'} = 0 \\ \text{(iii)} & \nabla_{A0'}\xi^A\eta^{1'} = 0 \\ \text{(iv)} & \nabla_{A1'}\xi^A\eta^{0'} = 0 \\ \text{(v)} & \nabla_{0B'}\xi^1\eta^{B'} = 0 \\ \text{(vi)} & \nabla_{1B'}\xi^0\eta^{B'} = 0 . \end{cases}$$

Combining (1.4) and (1.5) we obtain

(1.6) **Theorem** *There is a correspondence between (i) harmonic morphisms* $\phi : M \to \mathbf{C}$, $M \subset \mathbf{R}^4$, *and (ii) pairs of spinor fields* $(\xi^A, \eta^{A'})$ *on* M *satisfying the spinor equations:*

(1.7) $$\begin{cases} \nabla_{AA'}\xi^A\eta^{B'} = 0 \\ \nabla_{AB'}\xi^C\eta^{B'} = 0 . \end{cases}$$

Proof It is clear that (1.7) implies equations (1.4) and (1.5). Conversely, suppose we consider the first of equations (1.7) with $A' = B' = 0$. Then

$$\begin{aligned}\nabla_{00'}\xi^0\eta^{0'} + \nabla_{10'}\xi^1\eta^{0'} &= (\nabla_{00'}\xi^0\eta^{0'} + \nabla_{11'}\xi^1\eta^{1'} + \nabla_{10'}\xi^1\eta^{0'} + \nabla_{01'}\xi^0\eta^{1'})/2 \quad \text{by (1.5) (i) and (ii)} \\ &= 0 \quad \text{by (1.4).}\end{aligned}$$

The other equations are proved similarly.

Remark In terms of the geometric description of [PR]. At each point $x \in M$ where $\nabla_{AA'}\phi \neq 0$, $\xi_A(x)$ determines an α-plane $\alpha(x)$ on the quadric $Q_2 \subset \mathbf{CP}^3$, and $\eta_{A'}(x)$ determines a β-plane $\beta(x)$. Then $\alpha(x), \beta(x)$ intersect in a point of Q_2. This point corresponds (under the identification of Q_2 with the Grassmannian of oriented 2-planes in $\mathbf{R}^4$) to a real 2-plane through the origin in $\mathbf{R}^4$. This plane is the vertical space at x (the tangent to the fibre of ϕ through x), translated to the origin.

2. Examples

Particular examples of harmonic morphisms $\phi : \mathbf{R}^4 \to \mathbf{C}$ are given by maps which are holomorphic with respect to one of the Kähler structures on $\mathbf{R}^4$. Each Kähler structure arises from the standard one obtained by identifying $\mathbf{R}^4 \approx \mathbf{C} \times \mathbf{C}$, and composing with an isometry.

Use coordinates (z, w) for $\mathbf{C} \times \mathbf{C}$, so that $z = x^0 + ix^1$, $w = x^2 + ix^3$. Then

$\phi : M \to C$, $M \subset R^4$, is holomorphic if and only if

$$\frac{\partial \phi}{\partial \bar{z}} = \frac{\partial \phi}{\partial \bar{w}} = 0$$

if and only if

$$\nabla_{00'}\phi = \nabla_{10'}\phi = 0 .$$

Then clearly $\det \nabla_{AA'}\phi = 0$ and

$$\nabla_{AA'}\phi = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix} \begin{pmatrix} 0 & * \end{pmatrix}$$

so that $\eta_{B'} = (0 \ \ \lambda)$, for some $\lambda \in C$. Similarly ϕ is -holomorphic if and only if $\eta_{B'} = (\mu \ \ 0)$ for some $\mu \in C$. We now consider the effect of an isometry on the spinor decomposition of $\nabla_{AA'}\phi$.

There is a well known double cover $SU(2) \times SU(2) \to SO(4)$. Suppose that $\theta \in SO(4)$ and define $\tilde{\phi} = \phi \circ \theta$. Then $\nabla\tilde{\phi}(x) = \nabla\phi(\theta(x)) \circ \theta$. If $(A, B) \in SU(2) \times SU(2)$ covers θ, then the induced action on spinors is given by

$$\xi_A \eta_{A'} \to A\xi_A \eta_{A'} B^* ,$$

where $B^* = \bar{B}^T$, so that

$$(\xi_A, \eta_{A'}) \to (A\xi_A, \eta_{A'}B^*)$$

(i.e. $\tilde{\xi}_A(\theta(x)) = A(\xi_A(x))$ and $\tilde{\eta}_{A'}(\theta(x)) = (\eta_{A'}(x))B^*$). Note that under the equivalence $(\xi_A, \eta_{A'}) \sim (\lambda\xi_A, \eta_{A'}/\lambda)$, this is independant of the choice of (A, B) covering θ. In particular we see that $\phi : M \to C$ is $\pm$holomorphic with respect to a Kähler structure obtained from the standard one by an orientation preserving isometry if and only if $[\eta_{A'}] \in CP^1$ is constant. Similarly ϕ is $\pm$holomorphic with respect to a Kähler structure obtained by an orientation reversing isometry if and only if $[\xi_A] \in CP^1$ is constant. To summarize.

(2.1) Theorem *If $\phi : M \to C$, $M \subset R^4$ open, is a harmonic morphism, then ϕ is $\pm$holomorphic with respect to one of the Kåhler structures on R^4 if and only if either $[\eta_{A'}]$ or $[\xi_A]$ is constant.*

Another class of examples are those which have totally geodesic fibres. These are classified in [BW1]. If $\phi : M \to C$, M open in R^4, is a harmonic morphism with totally geodesic fibres, let N denote the leaf space of the fibres. Locally and in favourable circumstances globally, N can be given the structure of a smooth Riemann surface and ϕ is given implicitly by the equation

$$\alpha_0(\phi(x))x^0 + \alpha_1(\phi(x))x^1 + \alpha_2(\phi(x))x^2 + \alpha_3(\phi(x))x^3 = 1 ,$$

where $x = (x^0, x^1, x^2, x^3)$, $\alpha = (1/2h)(1 - f^2 - g^2, i(1 + f^2 + g^2), -2f, -2g)$ and $f, g, h : N \to C \cup \infty$ are meromorphic functions. In this case it is easily checked that ξ_A, $\eta_{A'}$ are given by

$$\xi_A = \frac{1}{\sqrt{\sqrt{2}\,(\alpha'.x)\ h}} \begin{pmatrix} f - ig \\ i \end{pmatrix}$$

$$\eta_{A'} = \frac{1}{\sqrt{\sqrt{2}\,(\alpha'.x)\,h}}\;(-\,if + g \qquad 1)$$

(here f, g and h are evaluated at $\pi(x)$, where π is projection onto N).

Note that in general neither of these are projectively constant and so the harmonic morphisms are not ±holomorphic. It is not known whether (1.7) has any solutions globally defined on $\mathbf{R}^4$ apart from those with $[\xi_A]$ or $[\eta_{A'}]$ constant, such solutions would define new harmonic morphisms from $\mathbf{R}^4$ to $\mathbf{C}$.

3. Interpretation in terms of twistor bundles

Here we relate our spinor description to the description given by the second author [W] in terms of twistor bundles, thus interpreting the equations (1.7) in terms of holomorphicity properties of Gauss sections. We briefly summarize the results of [W].

Let V be a 2-dimensional distribution in an oriented 4-dimensional Riemannian manifold M, and let H be the corresponding orthogonal 2-dimensional distribution. We may locally choose orientations for each $V_x, H_x, x \in M$, so that the combined orientation of $V_x \oplus H_x = T_xM$ is that of M. We then define almost complex structures J^V, J^H on each V_x, H_x to be rotation through $\pi/2$. Note that changing the orientation of V_x changes that of H_x and replaces (J^V, J^H) by $(-J^V, -J^H)$. All results below will be independant of this change, so that there is no loss of generality in assuming J^V, J^H are globally chosen.

The Gauss section of V, $\gamma : M \to G_2(TM)$ then maps into the Grassmannian of oriented 2-planes in TM. The almost complex structures J^V and J^H combine to give almost complex structures $J^1 = (J^V, J^H)$ and $J^2 = (-J^V, J^H)$ on each T_xM. Note that J^1 is compatible with the orientation, i.e. there exists an oriented basis of the form e_1, J^1e_1, e_2, J^1e_2, whereas J^2 is incompatible. Let Z^+ (resp. Z^-) be the fibre bundle over M whose fibre at x is all metric almost complex structures on T_xM which are compatible (resp. incompatible) with the orientation; these are the well-known twistor bundles of M [ES]. The distribution V defines section $\gamma^1 : M \to Z^+$ and $\gamma^2 : M \to Z^-$ by $\gamma^1(x) = J^1, \gamma^2(x) = J^2$ (where J^1, J^2 both act on T_xM). Note that if M is an open subset of Euclidean space $\mathbf{R}^4$, the twistor bundles are trivial $Z^\pm = M \times S^2$ and there is a well-known holomorphic bijection $G_2(\mathbf{R}^4) \approx S^2 \times S^2$.

Given a submersive harmonic morphism $M^4 \to$ surface, the tangent spaces to its fibres give an integrable, minimal and conformal distribution.

(3.1) **Theorem** [W] *Let* V *be a 2-dimensional distribution on a 4-dimensional Riemannian manifold* M. *Then* V *is integrable, minimal and conformal if and only if the section* $\gamma^1 : M \to Z^+$ *is holomorphic with respect to the almost complex structure* J^2 *on* M *and the section* $\gamma^2 : M \to Z^-$ *is -holomorphic with respect to the almost complex structure* J^1 *on* M.

Now let $\phi : M \to \mathbf{C}$ be a submersive harmonic morphism from an open subset M of $\mathbf{R}^4$. Then the tangent planes to the fibres determine a 2-dimensional distribution V on M. At each point x, V_x is given by

$$[\partial_3\phi, -\partial_2\phi, \partial_1\phi, -\partial_0\phi] \in Q_2 \subset \mathbf{CP}^3,$$

where $Q_2 \approx G_2(R^4)$ is the standard identification of the Grassmannian with the complex quadric. Then a direct computation verifies that $\gamma^1 = [\eta_{A'}]$ and $\gamma^2 = [\xi_A]$ in terms of the spinor decomposition $\nabla_{AA'}\phi = \xi_A\eta_{A'}$.

Write $W = \nabla\phi$, then

$$W^a \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}\begin{pmatrix} iW^0 + W^1 & W^2 + iW^3 \\ W^2 - iW^3 & iW^0 - W^1 \end{pmatrix} = \begin{pmatrix} \xi^0 \\ \xi^1 \end{pmatrix} (\eta^{0'} \quad \eta^{1'})$$

and at each point $x \in M$,

$$[W] = [-i(\xi^0\eta^{0'} + \xi^1\eta^{1'}), \xi^0\eta^{0'} - \xi^1\eta^{1'}, \xi^0\eta^{1'} + \xi^1\eta^{0'}, i(\xi^1\eta^{0'} - \xi^0\eta^{1'})] \in CP^3 .$$

But the standard identification $CP^1 \times CP^1 \to Q_2 \subset CP^3$ is given by

$$([\xi^0, \xi^1], [\eta^{0'}, \eta^{1'}]) \to [-i(\xi^0\eta^{0'} + \xi^1\eta^{1'}), \xi^0\eta^{0'} - \xi^1\eta^{1'}, \xi^0\eta^{1'} + \xi^1\eta^{0'}, i(\xi^1\eta^{0'} - \xi^0\eta^{1'})] .$$

Thus, identifying CP^1 with $C\cup\infty$ by stereographic projection $[\xi^0, \xi^1] \to \xi^0/\xi^1$ etc., we find

$$(3.2) \qquad \gamma^1 = \frac{W^2 + iW^3}{iW^0 - W^1} = \frac{iW^0 + W^1}{W^2 - iW^3}, \quad \gamma^2 = \frac{W^2 - iW^3}{iW^0 - W^1} = \frac{iW^0 + W^1}{W^2 + iW^3} .$$

Writing the spinor equations (1.7) in terms of W, they take the form

$$(3.3) \quad \begin{cases} \text{(i)} & (-i\partial_0 + \partial_1)(iW^0 + W^1) + (\partial_2 + i\partial_3)(W^2 - iW^3) = 0 \\ \text{(ii)} & (-i\partial_0 + \partial_1)(W^2 + iW^3) + (\partial_2 + i\partial_3)(iW^0 - W^1) = 0 \\ \text{(iii)} & (\partial_2 - i\partial_3)(iW^0 + W_1) + (-i\partial_0 - \partial_1)(W^2 - iW^3) = 0 \\ \text{(iv)} & (\partial_2 - i\partial_3)(W^2 + iW^3) + (-i\partial_0 - \partial_1)(iW^0 - W^1) = 0 \end{cases}$$

$$\begin{cases} \text{(v)} & (-i\partial_0 + \partial_1)(iW^0 + W^1) + (\partial_2 - i\partial_3)(W^2 + iW^3) = 0 \\ \text{(vi)} & (-i\partial_0 + \partial_1)(W^2 - iW^3) + (\partial_2 - i\partial_3)(iW^0 - W^1) = 0 \\ \text{(vii)} & (\partial_2 + i\partial_3)(iW^0 + W^1) + (-i\partial_0 - \partial_1)(W^2 + iW^3) = 0 \\ \text{(viii)} & (\partial_2 + i\partial_3)(W^2 - iW^3) + (-i\partial_0 - \partial_1)(iW^0 - W^1) = 0 . \end{cases}$$

Remark Of course these are equivalent to $\nabla_a W^a = 0$ and $\nabla_{[a}W_{b]} = 0$, expressing harmonicity and integrability respectively.

In order to show that equations (3.3) imply the holomorphicity results of Theorem (3.1), we consider a point x and suppose without loss of generality that ∂_2, ∂_3 span V_x. then $W^2 + iW^3 = W^2 - iW^3 = 0$ at x. Since the fibres of ϕ are minimal [BE], we also have $\partial_2W^2 + \partial_3W^3 = 0$ at x. In particular at x

$$(\partial_2 - i\partial_3)(W^2 + iW^3) = (\partial_2 + i\partial_3)(W^2 - iW^3) = 0$$

by minimality and integrability of the fibres.

Consider $\gamma^1 = (W^2 + iW^3)/(iW^0 - W^1)$. Then at x

$$(-i\partial_0 - \partial_1)(iW^0 - W^1) = 0$$

by (3.3)(viii). By horizontal conformality

$$(W^0 + iW^1)(W^0 - iW^1) = -(W^2 + iW^3)(W^2 - iW^3) .$$

So at x, either $W^0 + iW^1 = 0$ or $W^0 - iW^1 = 0$. Suppose $W^0 - iW^1 = 0$, in which case $W^0 + iW^1 \neq 0$. Then

$$(W^0 - iW^1)(\partial_2 + i\partial_3)(W^0 + iW^1) + (W^0 + iW^1)(\partial_2 + i\partial_3)(W^0 - iW^1) = 0$$

at x, so that

$$(\partial_2 + i\partial_3)(iW^0 + W^1) = 0$$

at x . Now (3.3)(vii) implies

$$(-i\partial_0 - \partial_1)(W^2 + iW^3) = 0,$$

so that

$$(-i\partial_0 - \partial_1)\gamma^1 = 0$$

and γ^1 is horizontally -holomorphic.

Writing $\gamma^1 = (iW^0 + W^1)/(W^2 - iW^3)$, a similar computation shows that $(\partial_2 + i\partial_3)\gamma^1 = 0$ and γ^1 is vertically holomorphic. Similarly γ^2 is horizontally -holomorphic and vertically -holomorphic. If on the other hand $W^0 + iW^1 = 0$, then the holomorphicity conditions are reversed. We have therefore shown directly that the spinor equations (1.7) give the equations of Theorem (3.1).

Conversely given a 2-dimensional distribution which is conformal, then it determines a null vector field which can be described by spinor fields $\xi_A, \eta_{A'}$. If the corresponding Gauss maps satisfy the holomorphicity equations of Theorem (3.1), then by that theorem the distribution is integrable and minimal and the spinor fields $\xi_A, \eta_{A'}$ satisfy equations (1.7).

This gives an interpretation for the spinor fields and equations of Theorem (1.6). The advantage of Theorem (1.6) over Theorem (3.1) is that it is valid for arbitrary harmonic morphisms (i.e. those with critical points).

4. Minkowski space

We consider a map $\phi : U \to \mathbf{C}$, U open in Minkowski space M^4, satisfying the equations:

(4.1) $$(\partial_0\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2 = 0$$

(4.2) $$\partial_0{}^2\phi - \partial_1{}^2\phi - \partial_2{}^2\phi - \partial_3{}^2\phi = 0 .$$

The spinor correspondence is given by

$$x^a \quad \leftrightarrow \quad x^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix}$$

Exactly as for the $\mathbf{R}^4$ case we obtain

(4.3) **Theorem** *There is a correspondence between*

(i) *mappings* $\phi : U \to \mathbf{C}$ *satisfying equations (4.1) and (4.2) and*

(ii) *pairs of spinor fields* $(\xi^A, \eta^{A'})$ *on* U *satisfying the spinor equations:*

$$\begin{cases} \nabla_{AA'}\xi^A\eta^{B'} = 0 \\ \nabla_{AB'}\xi^C\eta^{B'} = 0 . \end{cases}$$

5. The Euclidean $\mathbf{R}^3$ case

We define the spinor correspondence by

$$x^a = (x^1, x^2, x^3) \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} x^2 + ix^3 & -x^1 \\ -x^1 & -x^2 + ix^3 \end{pmatrix} = x^{AB}$$

Define the differential operators D_{AB} by

$$D_{00} = (\partial_2 - i\partial_3)/\sqrt{2}$$

$$D_{01} = -\partial_1/\sqrt{2}$$
$$D_{10} = -\partial_1/\sqrt{2}$$
$$D_{11} = (-\partial_2 - i\partial_3)/\sqrt{2}\,.$$

These agree with [S], equation (14). Let $M \subset \mathbf{R}^3$ be an open subset and suppose $\phi : M \to \mathbf{C}$ is a smooth mapping. Then, as for the $\mathbf{R}^4$ case, ϕ being horizontally conformal is equivalent to nullity of $D_{AB}\phi$ which is equivalent to

(5.1) $$D_{AB}\phi = \xi_A\xi_B$$

for some spinor field ξ_A on M.

If ϕ is horizontally conformal, so that $D_{AB}\phi = \xi_A\xi_B$, then ϕ is harmonic if and only if

(5.2) $$D_{AB}\xi^A\xi^B = 0$$

(if and only if $\xi^B D_{AB}\xi^A = 0$).

Conversely, as in [S], given a spacial null vector field $v = \mu^A\mu^B$, then curl v is given by $-i\sqrt{2}D^{C(A}\mu^{B)}\mu_C$. Combining this with equation (5.2) we obtain

(5.3) **Theorem** *There is a correspondence between harmonic morphisms* $\phi : M \to \mathbf{C}$, M *open in* $\mathbf{R}^3$, *and spinor fields* ξ^A *on* M *satisfying*

(5.4) $$D_{AB}\xi^A\xi^C = 0\,.$$

Note: A spinor field $\psi^{AB} = \xi^A\xi^B$ satisfying (5.4) may be interpreted as a null, source free, time independant solution to Maxwell's equations. This is in fact clear by expressing $\nabla\phi \equiv E + iB$ in real and imaginary parts. Then horizontal conformality implies $E.B = 0$, curl E = curl B = 0 is automatic and harmonicity gives div E = div B = 0 .

Given a harmonic morphism $\phi : M \to \mathbf{C}$, M open in $\mathbf{R}^3$, we can associate a Gauss map $\gamma : M \to S^2$, given by $\gamma(x)$ = unit positive tangent to the fibre of ϕ through x (see [B2, BW1]). In fact γ extends smoothly across critical points [BW3]. Then it is easily checked that in the chart given by stereographic projection $S^2 \to \mathbf{C}\cup\infty$, γ is represented by ξ_0/ξ_1. The equation (5.4) now has the simple interpretation of (i) minimality of the fibres, and (ii) horizontal holomorphicity of the Gauss map γ [B2, W].

Harmonic morphisms from open subsets of Euclidean space $\mathbf{R}^3$ have been completely classified in [BW1]. In fact locally ϕ is given implicitly by an equation

$$\alpha_1(\phi(x))x^1 + \alpha_2(\phi(x))x^2 + \alpha_3(\phi(x))x^3 = 1\,,$$

where $\alpha = (1/2h)(1 - g^2, i(1 + g^2), -2g)$ and h, g are meromorphic functions on a certain Riemann surface N (the leaf space of the corresponding folitation). In this case the corresponding spinor field is seen to be

(5.5) $$\xi_A = \frac{1}{\sqrt{\sqrt{2}\,\alpha'.x}}\left(\frac{1}{\sqrt{h}} \quad \frac{g}{\sqrt{h}}\right),$$

where g and h are functions of $\pi(x)$ where π is projection onto the leaf space N . By a result in [BW1], the only harmonic morphisms defined globally on $\mathbf{R}^3$ with values in a Riemann surface are given by an orthogonal projection followed by a weakly conformal map. In this case after appropriate choices of coordinates, $N \approx \mathbf{C}$, g is constant and $h(z) = z$. In particular this is true if and only if $[\xi_A]$ is constant.

Remark There is an interesting connection between harmonic morphisms $\phi : M \to C$, M open in $\mathbf{R}^3$, and solutions to the Bogomolny equations (magnetic monopoles). For both are classified in terms of holomorphic curves in the compex surface TS^2 [BW1, H]. For examples such as the axially symmetric solutions of Prasad and Rossi, the region of physical interest appears to be the envelope of the fibres of the harmonic morphism. These are precisely the points x where $\alpha'.x = 0$ [B2, BW1] and so correspond to the singularities of the spinor field given by (5.5).

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Invariants of CR Densities

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Invariants of Conformal Densities

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Abstract

This article is concerned with problems in parabolic invariant theory arising from flat conformal geometry. We show that such problems may be formulated in terms of the variational complex, taken from the formal theory of the calculus of variations. From known properties of this complex we are able to write down the general scalar differential invariant of functions in odd dimensions under conformal motions.

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