Twistors as Charges for Spin $\frac{3}{2}$ in Vacuum

In TN 31 (1990) pp. 6–8, I put forward the suggestion that it might be possible to develop a twistor theory for general vacuum space-times by describing a twistor as a kind of helicity $\frac{3}{2}$ elemental state. Elemental states were introduced by APH in TN 22 (1986), reprinted in F.A.T.T., vol 1 (1990). They provide a cohomological space-time interpretation of the internal lines $\circ\!\!-\!\!\bullet$ that make up twistor diagrams. For spin $\frac{3}{2}$, the elemental states would correspond to the internal lines "$\circ\overset{-2}{—}\bullet$" that occur with perturbative descriptions of gravitational scatterings, regarding "$W \overset{-2}{—} Z$" as a "twistor function" $f(W_\alpha)$ of homogeneity degree -1 describing a twistor Z^α as some kind of cohomological space-time field (of helicity $\frac{3}{2}$, since $f(W_\alpha)$ has homogeneity $1 = 2s-2 = 2\times\frac{3}{2}-2$). The idea here was to use the fact that spin $\frac{3}{2}$ fields are, in an appropriate sense, "consistent" in (and only in) Ricci-flat space-times (see H.A. Buchdahl, Nuovo Cim. 10 (1958) 96–103; S. Deser & B. Zumino, Phys. Lett. 62B (1976) 335–7; and especially B. Julia, Comptes Rendus (1982)), so the interpretation of twistors as helicity $\frac{3}{2}$ elemental states might also carry over to general Ricci-flat space-times. An essential problem had seemed to be how to characterize the particular degree 1 "twistor function $f(W_\alpha)$", in terms of its cohomological helicity $\frac{3}{2}$ space-time field that actually picks out $W \overset{-2}{—} Z$ amongst all twistor functions $f(W_\alpha)$. However, in flat space there is a special cohomological role for $W \overset{-2}{—} Z$, which can be reinterpreted in space-time terms, as providing Z^α as playing a role as a ("disembodied") charge for a helicity $\frac{3}{2}$ field.

There are various different formulations of the massless spin $\frac{3}{2}$ equations. The one I used in TN 3 might be called the "Dirac form" where, in flat space-time $\mathbb{M}$ we have the chain

$$\psi_{A'B'C'} = \psi_{(A'B'C')}\,, \quad \nabla^{AA'}\psi_{A'B'C'} = 0$$

$$\gamma^{A}_{B'C'} = \gamma^{A}_{(B'C')}\,, \quad \nabla^{BB'}\gamma^{A}_{B'C'} = 0\,, \quad \nabla_{AA'}\gamma^{A}_{B'C'} = \psi_{A'B'C'}$$

$$\rho^{AB}_{C'} = \rho^{(AB)}_{C'}\,, \quad \nabla^{CC'}\rho^{AB}_{C'} = 0\,, \quad \nabla_{BB'}\rho^{AB}_{C'} = \gamma^{A}_{B'C'}$$

with gauge freedom

$$\gamma^{A}_{B'C'} \longmapsto \gamma^{A}_{B'C'} + \nabla^{A}_{B'}\nu_{C'} \qquad \text{with} \quad \nabla^{AC'}\nu_{C'} = 0$$

$$\rho^{AB}_{C'} \longmapsto \rho^{AB}_{C'} + \varepsilon^{AB}\nu_{C'} + i\,\nabla^{A}_{C'}\chi^{B} \qquad \text{with} \quad \nabla_{AC'}\chi^{A} = 2i\,\nu_{C'}.$$

There is also a "gauge freedom of the second kind" (as in EM-theory) which leaves the potentials unchanged, given by

$$\nu_{A'} = \Pi_{A'}, \quad \chi^A = \Omega^A$$

where $(\Omega^A, \Pi_{A'})$ are the spinor parts of some twistor, i.e.

$$\nabla_{AA'}\Omega^B = -i\,\varepsilon_A{}^B\,\Pi_{A'} \quad \text{with } \Pi_{A'} \text{ constant.}$$

We have an exact sequence

$$0 \to \underset{(\Omega,\Pi)}{\text{gauge potls. of 2nd kind}} \to \underset{(\chi,\nu)}{\text{gauge quantities}} \to \underset{(\rho,\gamma)}{\text{potential}} \to \underset{(\psi)}{\text{field}} \to 0$$

In a general Ricci-flat curved space-time $\mathcal{M}$, most of this does not carry through, but we can still retain (Dirac form)

$$\gamma_{B'C'}^{A} = \gamma_{(B'C')}^{A}, \quad \nabla^{BB'}\gamma_{B'C'}^{A} = 0, \quad \text{with gauge } \nabla_{B'}^{A}\nu_{C'} \quad \text{where } \nabla^{AC'}\nu_{C'} = 0$$

as a consistent system (subject to some puzzling phenomena that we shall need to come back to later). (It is easy to see that the Ricci-flat condition is necessary and sufficient for the gauge freedom to be actually $\nabla_{B'}^{A}\nu_{C'}$; that it is sufficient also for the γ equations is more complicated.) People who work with supersymmetry, however, prefer what one may call the "Rarita-Schwinger form" where we do not impose symmetry $\gamma_{B'C'}^{A} = \gamma_{(B'C')}^{A}$, and take our equations as

$$\varepsilon^{B'C'}\nabla_{A(A'}\gamma_{B')C'}^{A} = 0, \quad \nabla^{B'(B}\gamma_{B'C'}^{A)} = 0, \quad \text{with gauge } \nabla_{B'}^{A}\nu_{C'} \text{ where } \nu_{C'} \text{ is unrestricted.}$$

The first of these equations asserts the total symmetry of $\psi_{A'B'C'} = \nabla_{A(A'}\gamma_{B')C'}^{A}$; however, it should be noted that if $\mathcal{M}$ is not a.s.d. (anti-self-dual), then ψ is not gauge-invariant (in either the Dirac or R-S form). The R-S form is easily reduced to the Dirac form by using a gauge choice for ν (solving an inhomogeneous Weyl neutrino equation) which makes γ symmetric, the remaining gauge freedom being subject to $\nabla^{AA'}\nu_{A'} = 0$.

In $\mathbb{M}$, we may obtain the charges for a helicity $3/2$ field by spin-lowering. Take the spinor field $\mu^{A'}$ to be the primary part of a dual twistor $W_\alpha = (\lambda_A, \mu^{A'})$:

$$\nabla_{AA'}\mu^{B'} = i\,\varepsilon_{A'}{}^{B'}\lambda_A, \quad \lambda_{A'} \text{ const.}$$

and set

$$\varphi_{A'B'} = \psi_{A'B'C'}\,\mu^{C'}$$

for some ψ in an open region $\mathcal{R}$ surrounding a worldtube. We can use the s.d. Maxwell field φ in a charge integral over some 2-surface $\mathcal{S}$ in $\mathcal{R}$ surrounding the tube:

$$\frac{i}{4\pi}\oint \varphi_{A'B'}\, dx_A^{A'} \wedge dx^{AB'} = \zeta(W) \quad = \text{electric} + i \times \text{magnetic}$$

where $\zeta(W)$ is linear in W_α and therefore $\zeta(W) = Z^\alpha W_\alpha$ for some Z^α. This Z^α is the required charge for ψ. Note that it does not depend on the particular choice of $\mathcal{S}$.

We want to do something here that has a chance of holding in $\mathcal{M}$ and hence of providing a definition of a twistor in general Ricci-flat space-times (and independent of any particular choice of $\mathcal{S}$). A possible suggestion would be to generalize the following, which works in $\mathbb{M}$: Consider fields ψ... which are global in $\mathcal{R}$ ($\subset \mathbb{M}$). We shall say that two such fields are equivalent if their difference has a (second) potential ρ (and therefore also a potential γ) which is global in $\mathcal{R}$. The space $\mathbb{T}^\alpha$ of twistors can then be identified with this space of equivalence classes

$$\mathbb{T}^\alpha = \{\text{global } \psi\text{s}\} \,/\, \{\text{global } \rho\text{s}\}.$$

We can also find the space $\mathbb{S}_{A'}$ of constant $\pi_{A'}$-spinors, and the space $\mathbb{S}^A$ of constant ω^A-spinors when $\pi_{A'} = 0$ as, respectively:

$$\mathbb{S}_{A'} = \{\text{global } \psi\text{s}\}/\{\text{global } \gamma\text{s}\}, \qquad \mathbb{S}^A = \{\text{global } \gamma\text{s}\}/\{\text{global } \rho\text{s}\}$$

illustrating
$$0 \to \mathbb{S}^A \to \mathbb{T}^\alpha \to \mathbb{S}_{A'} \to 0.$$

How much of this generalizes for a Ricci-flat $\mathcal{M}$? It may be remarked that if $\mathcal{M}$ is a.s.d. then ψ-fields exist and have the same relation to γs as in $\mathbb{M}$. This is consistent with the above and with the fact that $\mathbb{S}_{A'}$ is well-defined in the a.s.d. case. If $\mathcal{M}$ is s.d. then ρ-potentials exist (locally) and have the same relation to γs as in $\mathbb{M}$, consistent with the fact that $\mathbb{S}^A$ is well-defined in the s.d. case. In the a.s.d. case we can hope to go further and obtain the full curved twistor space $\mathcal{T}$ of the "non-linear graviton" construction. Although we do not have ρs locally, we can weaken the equation relating ρ to γ to

$$\gamma^A_{B'C'} = \nabla_{B(B'}\hat{\rho}^{AB}{}_{C')},$$

since this should be locally soluble, and then state the equivalence between two ψs in terms of the existence of a complex 2-surface (the α-surface of "Z") on which their respective $\hat{\rho}$s have an appropriate globality. This criterion seems too clumsy to be what is ultimately needed, but something of a non-linear nature is certainly required.

If $\mathcal{M}$ is general (Ricci-flat), or even merely s.d., we need a concept of a "global ψ" throughout $\mathcal{R}$ ($\subset \mathcal{M}$), even though the actual spinor ψ is not gauge invariant. In the case of $\mathcal{M}$, we can state such globality entirely in terms of γ by saying that $\mathcal{R}$ can be covered by open sets $\{\mathcal{U}_i\}$ where $\exists \overset{i}{\gamma}$ in $\mathcal{U}_i$ with

$$\overset{i}{\gamma} - \overset{j}{\gamma} = \nabla \overset{ij}{\nu} \quad \text{in each (non-empty) } \mathcal{U}_i \cap \mathcal{U}_j$$

where

$$\overset{ij}{\nu} \text{ exists in } \mathcal{U}_i \cap \mathcal{U}_j$$

with $\nabla^{AC'} \overset{ij}{\nu}_{C'} = 0$ (taking the Dirac form). We can then define

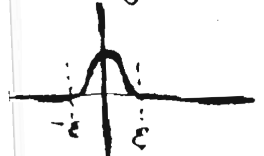
$$\overset{ijk}{\Pi} = \overset{ij}{\nu} + \overset{jk}{\nu} + \overset{ki}{\nu} \quad \text{in } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$$

and note that the Πs are constant since $\nabla \overset{ijk}{\Pi} = 0$. (We could also do the same with the ρs, χs and Ωs to obtain a twistor $(\overset{ijk}{\Omega}{}^{A}, \overset{ijk}{\Pi}_{A'})$ in $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$.) Appropriately adding these up, we obtain $(?\,\frac{1}{2\pi i}\times)\,\pi_{A'}$ (or $\frac{1}{2\pi i}(\omega^A, \pi_{A'})$) as required. This is an example of an "evaluation procedure" for a cohomology class — here an H^2 element, so we move 3 steps back from ψ in our exact sequence of potentials in order to get the charge for ψ.

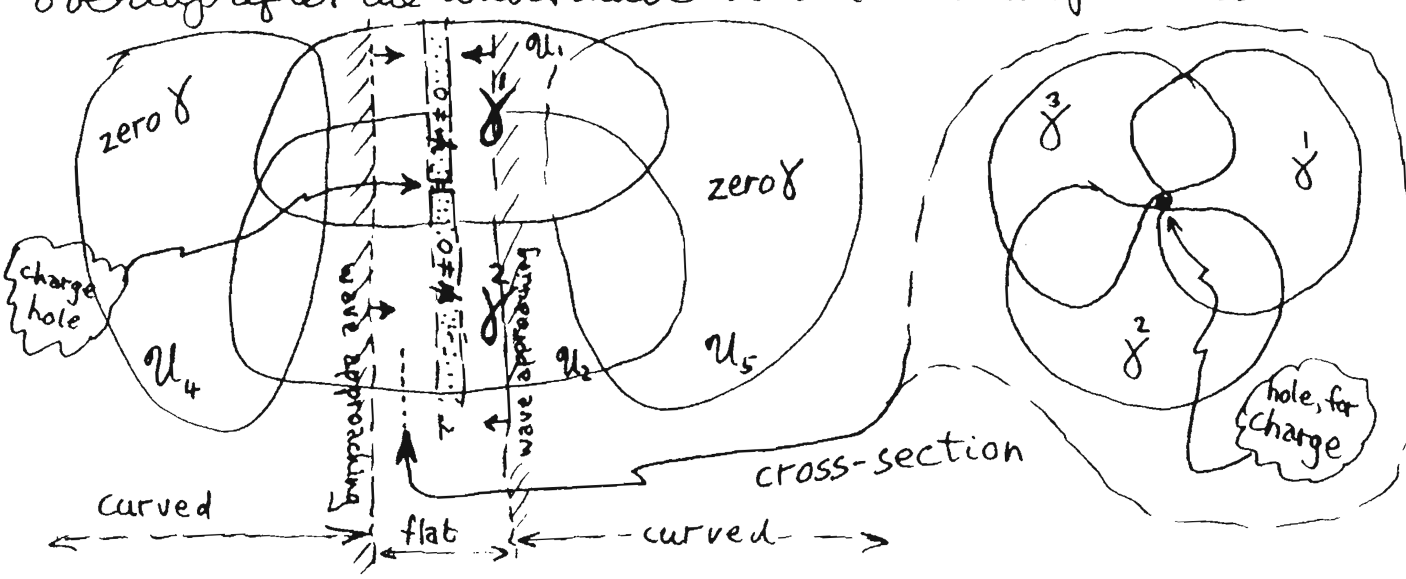
In a (non-a.s.d.) $\mathcal{M}$ something must go wrong with our original γs if we are to expect a non-zero $\pi_{A'}$-charge. For there are <u>no</u> constant $\Pi_{A'}$ spinors other than zero. This seems to be an almost paradoxical situation, because we can envisage situations where a "ψ" with a non-zero charge is set up in an initially flat region of space-time, and this "ψ" evolves, within its domain of dependence, into a full region of non-zero curvature. (Here "ψ" means γs modulo $\nabla\nu$s in a covering of $\mathcal{R}$, as described above.) We can arrange that the initial ψ-field is non-zero entirely within a thin slab, and singular only just around the centre of the slab, e.g. of the form

$$\zeta^{-1} o_{A'} o_{B'} o_{C'} B(u) \quad \text{or} \quad \bar{\zeta}^{-1} \iota_{A'} \iota_{B'} \iota_{C'} B(v)$$

at $u+v=0$ ("$t=0$") in a flat region with standard null coordinates $(u, v, \zeta, \bar{\zeta})$ and constant dyad (o^A, ι^A) for $\mathcal{M}$ (metric $ds^2 = 2\,du\,dv - 2\,d\zeta\,d\bar{\zeta}$, $\frac{\partial}{\partial v} = o^A \bar{o}^{A'} = \ell^a$, $m^a = \frac{\partial}{\partial \zeta}$, $\frac{\partial}{\partial u} = \iota^A \bar{\iota}^{A'} = n^a$) where B is a C^∞ "bump function", non-zero only in $(-\varepsilon, \varepsilon)$. Plane waves approach this region from either side:

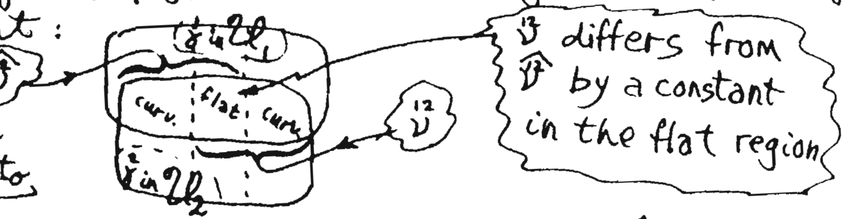


It would <u>seem</u> that we could cover a huge region of $\mathcal{M}$ with large open sets with large overlaps — large enough so that the domains of dependence of these open sets, individually, still overlap after the waves have collided and the space-time is

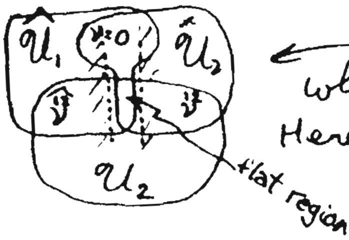


entirely curved. Plane waves have constant spinors but we can tilt them very slightly ~~so~~ that these constants are not in the directions of o^A or ι^A. Thus the $\pi_{A'}$-charges must disappear (these being in the directions of $o_{A'}$ or $\iota_{A'}$. How can this happen?

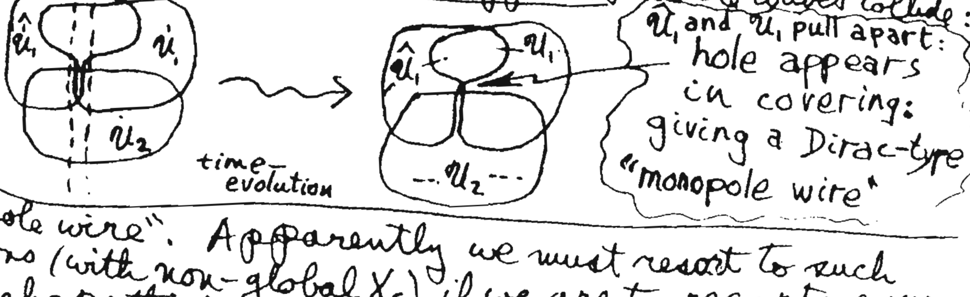
What actually occurs in this situation is somewhat surprising. It turns out that we find pairs of γs that are "locally" but not properly globally gauge equivalent:



To make this legitimate we must subdivide, say to



where $\hat{\gamma}$ and $\overset{1}{\gamma}$ are actually <u>identical</u> in $\hat{\mathcal{U}}_1 \cap \mathcal{U}_1$! Here, all 3 of $\hat{\gamma}$, $\overset{1}{\gamma}$ and $\overset{2}{\gamma}$ <u>are</u> properly gauge equivalent on their overlaps. However, the needed region of triple overlap now <u>disappears</u> as the waves collide:



It seems that the twistor-charge "escapes" along a Dirac-type "monopole wire". Apparently we must resort to such "wire-type" descriptions (with non-global γs) if we are to recapture our twistor concept. Perhaps this is no bad thing in the long run, since a <u>non-linear</u> (non-cohomological, in the strict sense) concept is needed. Work is in progress. There are further relations to twistor diagrams that will be reported on later.

Especial thanks go to K.P.T., who worked out the actual propagation of the above charged fields and potentials into plane-wave backgrounds — showing that we do not get a Hans Lewy obstruction (at least at this stage) as had seemed possible; also to L.J.M. for putting me straight on the R-S. equations. Comments from A.P.H., R.J.B. and T.S.-T. have also been valuable. — Roger Penrose