On the Topology of Quaternionic Manifolds

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Recall that a Riemannian manifold (M, g) of dimension $4k$, $k > 2$, is said to be *quaternionic-Kähler* if its holonomy group is a subgroup of $[Sp(k) \times Sp(1)]/Z_2$. Such manifolds are interesting, not only because they are automatically Einstein and occur in the holonomy classification of Riemannian geometries, but also because they have twistor spaces analogous to those familiar from dimension 4. Indeed, in the special case of $k = 1$, we define (M, g) to be *quaternionic-Kähler* iff it is self-dual Einstein.

The only known examples of compact quaternionic-Kähler manifolds of scalar curvature $R \neq 0$ are symmetric spaces, and indeed there are theorems [2, 4] asserting that a compact quaternionic-Kähler manifold of dimension 4 or 8 with $R > 0$ *must* be symmetric. While it remains unclear whether or not such a result might hold in higher dimensions, I would like to point out that, in this direction, one can at least say the following:

Theorem 1 *Let (M, g) be a compact quaternionic-Kähler $4k$-manifold with $R > 0$. Then either*
(a) $b_2(M) = 0$; or else
(b) M is the symmetric space $G_2(C^{k+2}) = SU(k+2)/S(U(k) \times U(2))$.

With a little bit of twistor theory, this is in fact an immediate consequence of some recent advances in algebraic geometry stemming from Mori's theory of extremal rays [3]. Let us begin by recalling that a compact quaternionic-Kähler $4k$-manifold with $R > 0$ has as its twistor space a compact complex $(2k+1)$-manifold Z which admits a complex contact form and a Kähler-Einstein metric with positive scalar curvature. In particular, $c_1 > 0$, so that Z is a so-called *Fano manifold*, and c_1 is divisible by $k + 1$. We can therefore invoke the following result of Wiśniewski [6]:

Theorem 2 *(Wiśniewski). Let X be a Fano manifold of dimension $2r-1$ for which $r|c_1$. Then $b_2(X)=1$ unless X is one of the following: (i) $CP_{r-1} \times Q_r$; (ii) $P(T^*CP_r)$; or (iii) CP_{2r-1} blown up along CP_{r-2}.*

Here $Q_r \subset CP_{r+1}$ denotes the r-quadric— that is, r-dimensional compactified complexified Minkowski space— while the projectivization of a bundle $E \to Y$ is defined by $P(E) := (E - 0_Y)/(C-0)$.

On the other hand, spaces (i) and (iii) aren't complex contact manifolds, as follows most easily seen from the fact that that $\Gamma(CP_{r-1}, \Omega^1(1)) = 0$, so that the obvious foliations by CP_{r-1}'s would necessarily have Legendrian leaves, so that these spaces would then have to be of the form $P(T^*Y)$, where Y is the leaf space Q_r or CP_r; contradiction. So the only candidate for a twistor space is (ii), which is indeed the twistor space of $G_2(C^{r+1})$. Theorem 1 now follows from the Leray-Hirsch theorem on sphere bundles: $b_2(Z) = b_2(M) + 1$.

In dimension 4, Theorem 1 contains Hitchin's result [2], since a compact self-dual manifold with $b_2 = 0$ must be conformally flat. On the other hand, there are already *two* symmetric examples with $b_2 = 0$ dimension 8, so the Poon-Salamon result [4] really contains further information about this case.

Is there a more elementary proof of Theorem 1? Haiwen Chen *(private communication)* has given a Weitzenböck argument for a weaker version of this theorem, but his proof unfortunately requires the hypothesis of positive *sectional* curvature.

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