Homogeneity of Twistor Spaces

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Let Z be a twistor space of a quaternionic Kähler manifold M of positive scalar curvature. If Z is compact and homogeneous, then it is known that M is symmetric [1]. In this note we will be concerned with the case when Z is not necessarily compact but is still homogeneous.

Such a twistor space is a complex contact manifold of dimension $2n+1$, so possesses a line bundle $\mathcal{L}$ and a holomorphic 1-form $\theta \in \Omega^1(\mathcal{L})$ such that $\theta \wedge (d\theta)^n$ vanishes nowhere. A holomorphic vector field on Z is an infinitesimal contact transformation if it preserves θ up to scale. We say Z is a homogeneous twistor space if the connected component G of the identity of the group of holomorphic contact transformations of Z acts transitively.

Lichnerowicz [2] shows that homogenous complex contact manifolds are related to coadjoint orbits. The construction is just derived from that for homogeneous symplectic spaces. Let H be the stabiliser of a point and let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. We have an exact sequence

$$0 \longrightarrow \ker\theta \longrightarrow T'Z \xrightarrow{\theta} \mathcal{L} \longrightarrow 0,$$

so θ^* gives an inclusion of $\mathcal{L}^*$ into $T'^*Z \cong \operatorname{Ann}\mathfrak{h} \subset \mathfrak{g}^*$. If π is the projection $\mathcal{L}^* \to Z$, then we may define a holomorphic 1-form α on $\mathcal{L}^*$ by

$$\alpha_b(v) = b(\theta(\pi_* v)) = (\theta^* b)(\pi_* v),$$

for $v \in T_b\mathcal{L}^*$ and $\omega = d\alpha$ is then a complex symplectic form on $\mathcal{L}^*$. Now any holomorphic contact transformation of Z lifts to a transformation preserving α and hence ω, so if $\tilde{X}$ is the lift of a vector field X we have

$$\tilde{X} \lrcorner \omega = \tilde{X} \lrcorner d\alpha = L_{\tilde{X}}\alpha - d(\tilde{X} \lrcorner \alpha) = -d(\tilde{X} \lrcorner \alpha),$$

showing that the immersion $\mathcal{L}^* \to \mathfrak{g}^*$ is just a moment map, up to sign. Note that the moment map commutes with action of the scalars on the fibres of $\mathcal{L}^*$. Now, in general, if μ^X is a moment map for a vector field X, then the moment map $\mu^{[X,Y]}$

of the bracket of two vector fields differs from the Poisson bracket $\{\mu^X, \mu^Y\}$ by a constant. However, in our case, both these expressions are linear functions fibrewise, so the constant must be zero (cf. [3]). Thus the image of the moment map is a union of orbits for coadjoint action of G on $\mathfrak{g}^*$.

In the case of homogeneous complex contact manifolds there are now two cases: either G acts transitively on $\mathcal{L}^* \setminus 0$ or it does not. However, if Z is a twistor space then $\mathcal{L}^* \setminus 0$ is a hyperKähler manifold whose tangent space contains the lift of $\ker \theta$ as quaternionic subspace [4]. For a given point of $\mathcal{L}^* \setminus 0$ we may now use the action of the complex structures J and K on the tangent space of $\mathcal{L}^* \setminus 0$ to obtain a holomorphic vector field generating the action of the scalars through that point, so the image of the moment map consists of just one orbit (which contains zero in its closure).

If G is reductive, then we may choose an invariant inner product and identify $\mathfrak{g}^*$ with $\mathfrak{g}$. The image of $\mathcal{L}^* \setminus 0$ is now a nilpotent orbit in the semi-simple part of $\mathfrak{g}$. However, it was shown in [4], that the projectivisation of a nilpotent orbit is the twistor space of a quaternionic manifold of positive scalar curvature. To summarise:

If Z is a twistor space of a quaternionic Kähler manifold of positive scalar curvature such that Z is homogeneous as a complex contact manifold and the symmetry group G is reductive, then Z is the projectivisation of a nilpotent orbit of the semi-simple part of $\mathfrak{g}$, up to finite covers.

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References

[1] Boothby, W.M., 1961, *Homogeneous Complex Contact Manifolds,* Proc. Symp. Pure Math., **3**, 144–154; 1962, *A Note on Homogeneous Compex Contact Manifolds,* Proc. AMS, **13**, 276–280; Wolf, J.A., 1965, *Complex Homogeneous Contact Manifolds and Quaternionic Symmetric Spaces,* J. Math. Mech., **14**, 1033–1047.

[2] Lichnerowicz, A., 1986, *Espaces homogènes de contact complexes et espaces homogènes localemant conformément symplectiques complexes hamiltoniens,* C.R. Acad. Sci. Paris, **302**, 395–399; 1988, *Variétés de Jacobi et espaces homogènes de contact complexes,* J. Math. Pures Appl., **67**, 131–173.

[3] Arnol'd, V.I., "Mathematical Methods of Classical Mechanics," GTM **60**, Springer.

[4] Swann, A.F., 1991, *HyperKähler and Quaternionic Kähler Geometry,* to appear in Math. Annalen.