SL(2)_g = Spin Networks
by Louis H. Kauffman

The original spin networks of Roger Penrose [3] are based on SL(2)-invariant tensors. As explained in [1], the binor calculus can be seen as a special case of the bracket model of the Jones polynomial. In fact, this viewpoint extends to a generalization of spin nets corresponding to the re-coupling theory for the quantum group SL(2)_g. (See [2].) It is the purpose of this note to indicate the basic ingredients in this generalization.

First, g = \sqrt{A} as in [D]. We replace the binor identity with the bracket identity

\[ X = A \otimes + A^{-1} \otimes \]

and loop values \( O = -A^2 - A^{-2} = \delta \). The result of expanding any link diagram is then its bracket polynomial. For example

\[ \infty = A \otimes + A^{-1} \otimes \]

\[ = (A + A^{-1}(-A^2 - A^{-2})) \delta \]

\[ \infty = -A^{-3} \delta \]

and

\[ \otimes \otimes = A^{-1} \otimes + A^+ \otimes \]

\[ = A^{-1}(-A^3) \delta + A^+(-A^3) \delta \]

\[ \otimes \otimes = (-A^{-4} - A^4) \delta \].

Note the convention for using this generalized binor identity:

Turn the over-crossing line counterclockwise. Label the regions swept out as A. Label the remaining two regions as B = A^\text{-1}. In the expansion...
\[
\frac{\overline{AB}}{\overline{BA}} = A(\overline{A} + B)(\overline{A} - B)
\]

Evaluations of link diagrams via the bracket identity are invariant under regular isotopy: \(\overline{\ddagger_1} \approx \overline{\ddagger_2} \), \(\overline{\ddagger_3} \approx \overline{\ddagger_4} \), and satisfy the following rules for curls: \(\overline{\ddagger} = (\overline{A^3}) \), \(\overline{\ddagger} = (\overline{A^3}) \).

Now, replace the anti-symmetrizer by
\[
\frac{1}{n} = \sum_{\sigma \in \mathfrak{S}_n} (\tau A^3)^{\tau(\sigma)} \overline{\sigma}
\]

where \(\overline{\sigma}\) is a minimal braid projecting to the permutation \(\sigma\) (all crossings of type \(A \cap X\)), and \(\tau(\sigma)\) denotes the minimal number of transpositions required to write \(\sigma\).

For example:
\[
\frac{1}{2} = \frac{1}{2} + A^{-3} X
\]
\[
\frac{1}{3} = \frac{1}{3} + A^{-3} X + A^{-3} X + A^{-6} X + A^{-6} X + A^{-9} X
\]

Note that at \(A = -1\) we recover the binor identity, and the usual anti-symmetrizers. The basic property of an anti-symmetrizer is that 1) \(\overline{\frac{1}{n}} = 0\)

2) \(\frac{\frac{1}{n}}{\frac{1}{n}} = n! \)


Here both properties are true, with an appropriate generalization of the factorial:
\[ n! = \sum_{\sigma \in S_n} (A^{-\sigma})^T (\sigma) = \prod_{k=1}^n \frac{1-A^{-k}}{1-A^{-x}}. \]

Note how this works:
\[ U + A^{-3} X = U + A^{-3} (-A^3) U = 0. \]

Having defined anti-symmetrizers, we now can define 3-vertices:
\[ a \triangleleft b \triangleleft c \overset{\text{def}}{=} a \otimes i \otimes j \otimes k \triangleleft c \quad \text{admissible when } a, b, c \text{ satisfy the triangle inequality, and } a+b+c \text{ is even}. \]

The usual apparatus of recoupling theory now generalizes and each quantity can be expressed as a \( g \)-spin network evaluation. For example,
\[ r \otimes s \otimes a = u \otimes t \otimes s \overset{\text{def}}{=} u = \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \]

and \[ r \otimes s \otimes a = \emptyset \text{ if } a \neq b \quad (\emptyset \text{ denotes zero}). \]

Thus we can define \( g \)-6j symbols via the recoupling:
\[ b \triangleright j \triangleright c \bigtriangleup d = \sum_{i \neq j} \{ a \ b \ i \ c \ d \ j \}. \]
Then, taking traces, we find
\[
\left\{ \begin{array}{c}
 a \\
 b \\
 c \\
 d \\
 i
\end{array} \right\} = \frac{1}{\sqrt{2}} \left[ \begin{array}{c}
 b \\
 a \\
 d \\
 c
\end{array} \right] \left[ \begin{array}{c}
 \gamma_i \\
 0 \\
 0 \\
 0
\end{array} \right].
\]

Specific formulas for these \(8-6j\) coefficients can then be obtained just as in the
chromatic method for classical spin networks.

Finally, it is worth noting that each
anti-symmetric tensor can be written in
expanded form as an element in the
Temperley-Lieb algebra generated by
\[
\begin{array}{cccc}
\mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3
\end{array}
\]

For example:
\[
\mathbf{11} = \mathbf{11} + A^{-3} \mathbf{X} = \mathbf{11} + A^{-3} (A \mathbf{U} + A^{-1}) (\mathbf{11})
\]
\[
= \mathbf{11} + A^{-3} (A \mathbf{e}_1 + A^{-1})
\]
\[
= (\mathbf{11} + A^{-4}) + A^{-2} \mathbf{e}_1
\]
\[
= (\mathbf{11} + A^{-4}) [1 + A^{-1} \mathbf{e}_1]
\]
\[
= (2!) [1 + A^{-1} \mathbf{e}_1].
\]

In general, \(\mathbf{11}^n = (2!)^n \mathbf{e}_n\) where \(\mathbf{e}_i \cdot \mathbf{e}_n = 0\) for \(i \leq n\) and \(\mathbf{e}_n^2 = \mathbf{e}_n\).

In the next installment, we shall
discuss the role of the Spin Geometry
Theorem in this context.
References


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Appendix: explicit coordinate expressions

One of the main features of the Kozameh-Newman formalism is the use of the 4 functions $Z^{AA'} = \partial^A \partial^{A'} Z$ at some fixed $\pi^{AA'}$, $\pi_A$ as coordinates on $M$. The 0th and 1st derivatives of $Z$ are determined from $Z^{AA'}$ by $Z = \pi^A \pi_A Z^{AA'}$ and $Z^{AA'} = \pi^A Z^{AA'}$ (these follow from the homogeneity of $Z$) so $Z^{AA'}$ is the part of the second jet of $Z$ as a function of $\pi^{AA'}$, $\pi_A$ containing only the mixed second derivatives. In flat space $Z$ can be taken to be $Z = x^{AA'} \pi_A \pi_A$ where $x^{AA'}$ are affine coordinates on Minkowski space, so $Z^{AA'} = x^{AA'}$.

Note that if a quantity $f$ has homogeneity $n$ in $\pi^{AA'}$, then $\pi^A \partial^{A'} \partial^{A''} \cdots \partial^{A'''} f = 0$ by homogeneity so that $\partial^{A'} \partial^{A''} \cdots \partial^{A'''} f = \pi^{A'} \pi^{A''} \cdots \pi^{A'''} \partial^{A'''} f$ for some quantity $\partial^{A''} f$ of weight $n - 2$. Transferring $\Lambda = \delta^2 Z$, and $\Lambda^{AA'} = \delta^{A} \partial^{AA'}$ to the $Z^{AA'}$ coordinate system, we find that (*) becomes:

$$
\begin{align*}
0 &= g^{A(A'B')}B + \pi^A \pi^{A'} \delta^2 \pi^A \pi^{B'} \partial_{\xi} (\Lambda^{AA'}B^{B'})D_{\xi} + \pi^{A'} \pi^B \pi^{A'} \delta^2 \partial_{\xi} (\Lambda^{AA'}B^{B'})D_{\xi} \\
&+ \frac{1}{2} \pi^{A'} \pi^{A} \pi^{B'} \pi^{B} (\pi^{CC'} \partial_{\xi} \partial^2 \pi^{CC'} \partial_{\xi} g^{cc} + \partial_{\xi} \partial_\xi (\pi^A \partial^{AA'} A^{AA'})) \delta^2 \\
&\text{where } \xi, \partial_{\xi} \text{ are the concrete indices associated to the } Z^{AA'} \text{ coordinate system; } \xi = CC' \text{ etc.} \quad \text{If we adjoin to this the equation } g^{A(A'B')B} = \Omega^2 \varepsilon^{AB} \varepsilon^{A'B'} \text{ where } \Omega \text{ is the undetermined conformal factor, one can solve for } g^{ab} \text{ provided } (\pi^{AA'} \partial^{AA'} A^{AA'} \pi^A \partial_{AA'}) \neq 1.
\end{align*}
$$