$SL(2)_q$ – Spin Networks

by Louis H. Kauffman

The original spin networks of Roger Penrose [3] are based on $SL(2)$ – invariant tensors. As explained in [1], the binor calculus can be seen as a special case of the bracket model of the Jones polynomial. In fact, this viewpoint extends to a generalization of spin nets corresponding to the re-coupling theory for the quantum group $SL(2)_q$. (See [2].) It is the purpose of this note to indicate the basic ingredients in this generalization.

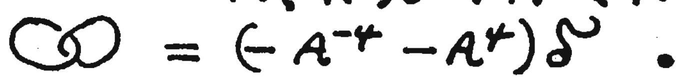
First, $q = \sqrt{A}$ as in [1]. We replace the binor identity with the bracket identity

and loop value $\bigcirc = -A^2 - A^{-2} = \delta$. The result of expanding any link diagram is then its bracket polynomial. For example

$$= (A + A^{-1}(-A^2 - A^{-2}))\delta$$

and

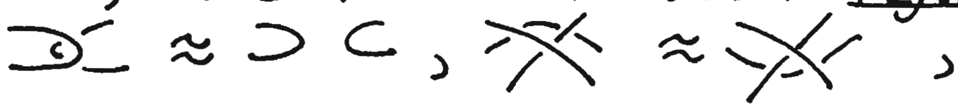
$$= A^{-1}(-A^{-3})\delta + A^{+1}(-A^3)\delta$$



Note the convention for using this generalized binor identity:

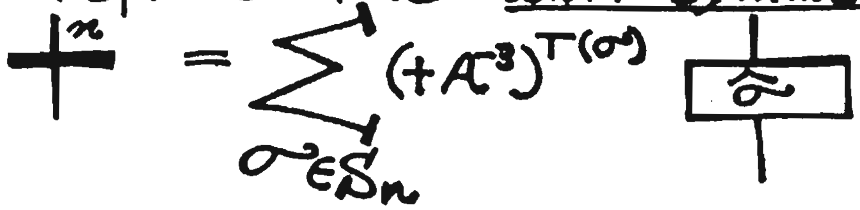
Turn the over-crossing line counterclockwise. Label the regions swept out as A. Label the remaining two regions as $B = A^{-1}$. In the expansion

Evaluations of link diagrams via the bracket identity are invariant under <u>regular isotopy</u>:



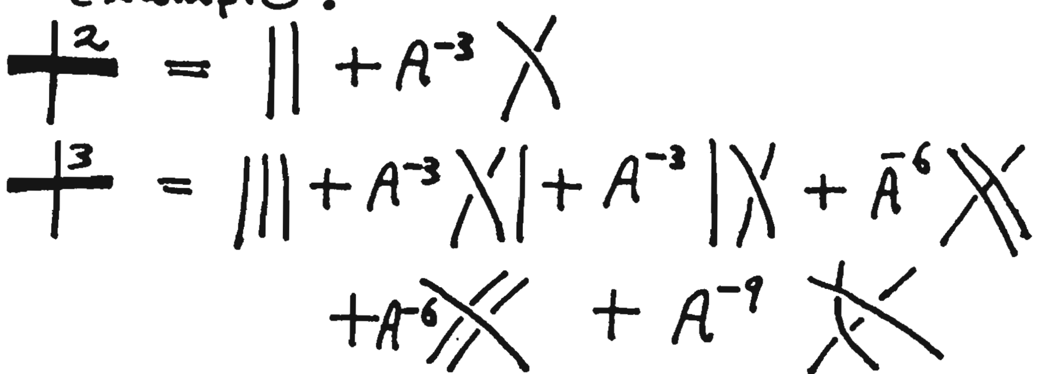
and satisfy the following rules for curls:

<u>Now</u>, replace the <u>anti-symmetrizer</u> by

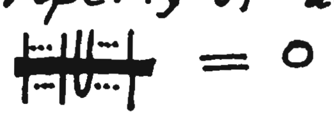


where $\hat{\sigma}$ is a minimal braid projecting to the permutation σ (all crossings of type A: X), and $T(\sigma)$ denotes the minimal number of transpositions required to write σ.

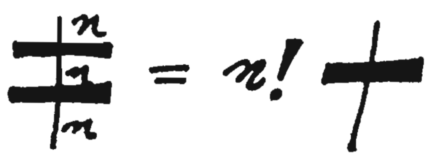
For example:



Note that at $A=-1$ we recover the binor identity, and the usual anti-symmetrizers.
The basic property of an anti-symmetrizer is that 1)



2)

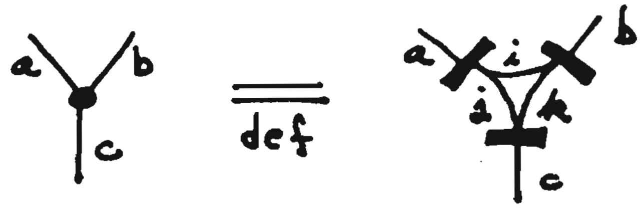


Here both properties are ~~true~~, with an appropriate generalization of the factorial:

$$n! = \sum_{\sigma \in S_n} (A^{-4})^{T(\sigma)} = \prod_{k=1}^{n} \left(\frac{1 - A^{-4k}}{1 - A^{-4}} \right).$$

Note how this works:

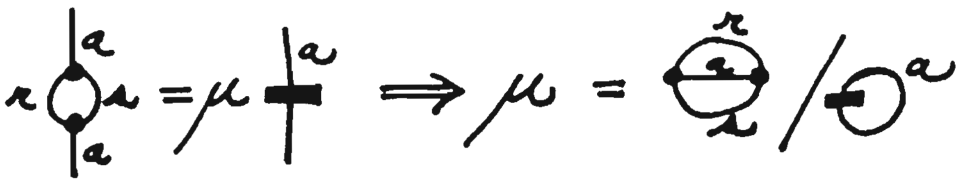
Having defined anti-symmetrizers, we now can define 3-vertices:

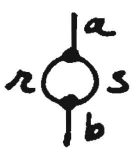


$i + j = a$
$j + k = c$
$i + k = b$,

admissible when a, b, c satisfy the triangle inequality, and a+b+c is even.

The usual apparatus of recoupling theory now generalizes and each quantity can be expressed as a <u>q-spin network evaluation</u>. For example,

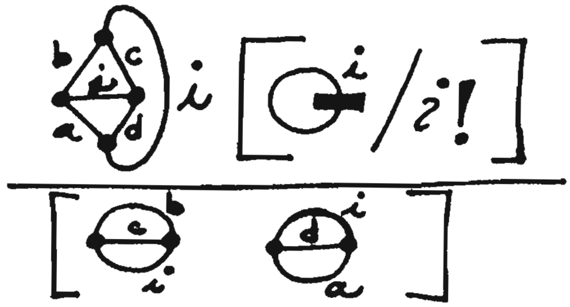


and  $= \phi$ if $a \neq b$ (ϕ denotes zero.).

Thus we can define <u>q-6j symbols</u> via the re-coupling

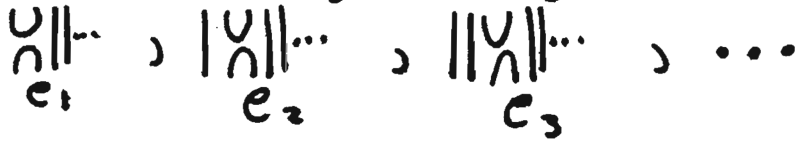
Then, taking traces, we find

$$\left\{ \begin{matrix} a & b & i \\ c & d & j \end{matrix} \right\} =$$

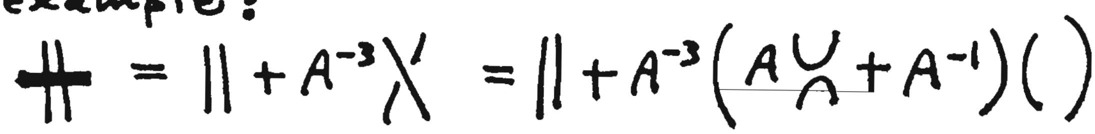


Specific formulas for these q-$6j$ coefficients can then be obtained just as in the chromatic method for classical spin networks.

Finally, it is worth noting that each anti-symmetrizer can be written in expanded form as an element in the Temperley-Lieb algebra generated by



For example:



$$= 1 + A^{-3}(Ae_1 + A^{-1})$$

$$= (1 + A^{-4}) + A^{-2}e_1$$

$$= (1 + A^{-4})[1 + \delta^{-1}e_1]$$

$$= (2!)[1 + \delta^{-1}e_1].$$

In general,  $= (n!)f_n$ where

$e_i f_n = 0$ for $i \le n$ and $f_n^2 = f_n$.

In the next installment, we shall discuss the fate of the Spin Geometry Theorem in this context.

References

1. L. Kauffman, Spin networks and the Jones polynomial. Twistor Newsletter No. 29, November 1989.
2. L. Kauffman, Knots, spin networks and 3-manifold invariants. (to appear).
3. R. Penrose. Angular momentum – an approach to combinatorial space-time. from Quantum Theory and Beyond ed. by T. Bastin. Camb. Univ. Press (1979).

Dept. of Math. Stat. Comp. Sci.
Univ. of Ill. at Chicago
Chicago, Ill. 60680

L. Mason: A new programme for light cone cuts and Yang–Mills holonomies

Appendix: explicit coordinate expressions

One of the main features of the Kozameh-Newman formalism is the use of the 4 functions $Z^{AA'} = \partial^A\partial^{A'}Z$ at some fixed $\pi_{A'}$, $\bar\pi_A$ as coordinates on $\mathcal{M}$. The 0^{th} and 1^{st} derivatives of Z are determined from $Z^{AA'}$ by $Z = \pi_{A'}\bar\pi_A Z^{AA'}$ and $Z^{A'} = \bar\pi_A Z^{AA'}$ (these follow from the homogeneity of Z) so $Z^{AA'}$ is the part of the second jet of Z as a function of $\pi_{A'}$, $\bar\pi_A$ containing only the mixed second derivatives. In flat space Z can be taken to be $Z = x^{AA'}\pi_{A'}\bar\pi_A$ where $x^{AA'}$ are affine coordinates on Minkowski space, so $Z^{AA'} = x^{AA'}$.

Note that if a quantity f has homogeneity n in $\pi_{A'}$, then $\pi_{A'}\partial^{A'}\partial^{A'_1}\ldots\partial^{A'_n}f = 0$ by homogeneity so that $\partial^{A'}\partial^{A'_1}\ldots\partial^{A'_n}f = \pi^{A'}\pi^{A'_1}\ldots\pi^{A'_n}\eth^{n+1}f$ for some quantity $\eth^{n+1}f$ of weight $-n-2$. Transfering $\Lambda = \eth^2 Z$, and $\Lambda^A = \partial^A\eth^2 Z$ to the $Z^{AA'}$ coordinate system, we find that $(*)$ becomes:

$$0 = g^{A(A'B')B} + \bar\pi^A\bar\pi^B\bar\pi_D\partial_{\underline{c}}\bar\Lambda^{(A'}g^{B')D\underline{c}} + \pi^{A'}\pi^{B'}\pi_{D'}\partial_{\underline{c}}\Lambda^{(A}g^{B)D'\underline{c}}$$

$$+\tfrac{1}{2}\pi^{A'}\pi^{B'}\bar\pi^A\bar\pi^B(\pi_{C'}\bar\pi_C\partial_{\underline{d}}\eth^2\bar\eth^2 Z + \partial_{\underline{c}}\Lambda\partial_{\underline{d}}\bar\Lambda)g^{\underline{cd}}$$

where $\underline{c}$, $\underline{d}$ are the concrete indices associated to the $Z^{AA'}$ coordinate system; $\underline{c} = CC'$ etc.. If we adjoin to this the equation $g^{A[A'B']B} = \Omega^2\varepsilon^{AB}\varepsilon^{A'B'}$ where Ω is the undetermined conformal factor, one can solve for g^{ab} provided $(\pi^{A'}\bar\pi^A\partial_{AA'}\Lambda\ \pi^{A'}\bar\pi^A\partial_{AA'}\bar\Lambda) \neq 1$.