A new programme for light cone cuts and Yang-Mills holonomies

In a series of papers Kozameh & Newman have developed formalisms for the study of general space-times and Yang-Mills fields that generalize many of the ingredients of the nonlinear graviton construction and Ward construction to the non self-dual equations. An important motivation for the study of these formalisms is that it seems likely that these structures will have to play a role in any twistorial understanding of the full vacuum and Yang-Mills equations even if only as an intermediate stage in some more elaborate framework.

To date, however, the incorporation of the full vacuum equations into the light cone cut formalism has been problematic even in principle and it has only been possible to do so by introducing substantial additional structure (the holonomy of one of the spin connections as an additional separate connection satisfying its own equations) Kozameh, Lamberti & Newman (1990). The purpose of this note is to present a new strategy that gives a clean articulation of the vacuum equations and more generally the Bach equations within the framework. The articulation of the field equations reduces to one scalar differential equation (and some boundary conditions). Unfortunately the calculations required to obtain an explicit form for the main equation have so far proved intractable except in the linearized and self-dual case. Nevertheless certain basic features are clear from its derivation and linearized form.

The basic field equations for the Yang-Mills formalism have already been obtained in two forms in Kozameh & Newman (1985). However, the new strategy yields a new point of view on these equations that eliminates various steps in Kozameh & Newman's approach and shows that only one scalar equation needs to be considered. The Bach equations are in fact the Yang-Mills equations for the local twistor connection so there is more contact between the two formalisms than one might have originally supposed.

The proposed strategies are not yet fully worked out and in the fully nonlinear cases various difficulties remain, so the basic idea will be illustrated by the linearized version of the theory where the ideas all work. In this note I shall use homogeneous coordinates that lead to simplifications in the derivation of many of the formulae. In the case of gravitation I also derive the formalism relative to a space-like hypersurface—asymptotic simplicity is inessential.

§1 The basic formalism. The formalism for curved space-times is based on an identification of the space of scaled null geodesics $\mathcal{N}$ with $T^*\mathcal{O}(1,1)$, where $\mathcal{O}(1,1)$ is the line bundle of homogeneity degree $(1,1)$ functions on the sphere. In the usual presentation of the formalism, $\mathcal{O}(1,1)$ is past null infinity and a scaled null geodesic is represented by a 1-form orthogonal to it at its intersection point with null infinity. The identification can also be made by identifying the light cone of a finite point of $\mathcal{M}$ with $\mathcal{O}(1,1)$ and $\mathcal{N}$ with its cotangent bundle.

The bundle $\mathcal{O}(1,1)$ is coordinatized by $(u,\pi_{A'},\bar\pi_A)$ which are taken to be homogeneous coordinates, $(u,\pi_{A'},\bar\pi_A)\sim(\lambda\bar\lambda u,\lambda\pi_{A'},\bar\lambda\bar\pi_A)$, $\pi_{A'}\neq 0$. If $u=Z(\pi_{A'},\bar\pi_A)$ is a section of $\mathcal{O}(1,1)$, then it determines the cotangent vector $(\partial^{A'}Z)d\pi_{A'}+(\partial^A Z)d\bar\pi_A$ where $\partial^{A'}=\partial/\partial\pi_{A'}$, $\partial^A=\partial/\partial\bar\pi_A$ and homogeneity implies that $Z=\pi_{A'}\partial^{A'}Z=\bar\pi_A\partial^A Z$. The cotangent bundle of $\mathcal{O}(1,1)$ can therefore be coordinatized by $(Z^A,\pi_{A'},Z^{A'},\bar\pi_A)$ subject to the relation $\pi_{A'}Z^{A'}=\bar\pi_A Z^A$. (These coordinates, strictly speaking, involve an additional phase—we are really dealing with the space of null geodesics with a complex scale, that is a parallel propagated spinor aligned along the null geodesic rather than just a covector.)

Remark: One can also produce such an identification relative to a spacelike hypersurface $\mathcal{H}$. All the spinor indices in the following should be taken to be concrete. Choose coordinates $x^{AA'}$ on a space-time $\mathcal{M}$ such that for some fixed $T_{AA'}$, $\mathcal{H}$ is given by $T_{AA'}x^{AA'}=0$. Let $\pi_{A'}$ be homogeneous coordinates on $\mathbb{CP}^1$. Define $Z(x,\pi_{A'},\pi_A)=x^{AA'}\pi_{A'}\bar\pi_A$. The space of scaled null geodesics is canonically identified with $T^*\mathcal{H}$. We define a map from $T^*(\mathcal{O}(1,1))$ to $T^*\mathcal{H}$ by identifying the point $(Z^A,\pi_{A'},Z^{A'},\pi_A)$ in $T^*(\mathcal{O}(1,1))$ with the cotangent vector $\pi_{A'}\bar\pi_A dx^{AA'}$ at the point $x^{AA'}$ whose coordinates are determined by the relations $T_{AA'}x^{AA'}=0$, $x^{AA'}\pi_{A'}=Z^A$ and $x^{AA'}\bar\pi_A=Z^{A'}$ (these are only 4 relations as $\pi_{A'}Z^{A'}=\bar\pi_A Z^A$). The coordinates $(Z^A,\pi_{A'},Z^{A'},\pi_A)$ can be thought of as being coordinates on the spin bundle of $\mathcal{M}$ restricted to $\mathcal{H}$ (although for this one must choose an arbitrary identification of phase of $\pi_{A'}$ with that of the spinors) and can be continued to be functions on the spin bundle of $\mathcal{M}$ by requiring that they be constant along the null geodesic spray. Note that the identification of the $\pi_{A'}$ with the spinors at each point cannot be made to be holomorphic without substantial restriction on the curvature.

Clearly, such an identification encodes little of the space-time geometry. The conformal structure is encoded by the knowledge, for each x in $\mathcal{M}$, of the S^2 in $P\mathcal{N}$ of light rays incident with x. This can be represented by a 'cut function' $Z(x,\pi,\bar\pi)$ which yields the S^2 of light rays incident with x as $(Z^A,\pi_{A'},Z^{A'},\bar\pi_A)=(\partial^A Z,\pi_{A'},\partial^{A'}Z,\bar\pi_A)$ in $T^*(\mathcal{O}(1,1))$. The cut function should be thought of as a generating function for these S^2's—the S^2's are always regular, whereas Z may well be singular. The cut function is the basic variable in the formalism.

§2 The reconstruction of the conformal structure from the cut function. This relies only on the property that, for each $\pi_{A'}$, Z is constant along a foliation by null hypersurfaces and that, for fixed x, as $\pi_{A'}$ varies, these surfaces vary through all null hypersurface elements at x (this follows from the fact that holding Z and $\pi_{A'}$ and $\bar\pi_A$ constant gives a Legendrian submanifold in $P\mathcal{N}$ and hence a null hypersurface in $\mathcal{M}$). Let ∂_a denote the coordinate derivative on $\mathcal{M}$ holding $\pi_{A'}$ constant. Then the conformal metric on 1-forms is determined by the condition $g^{ab}\partial_a Z\,\partial_b Z = 0$ as $\pi_{A'}$ and $\bar\pi_A$ vary. (Here the indices $a,b\ldots$ are regular tangent space indices to $\mathcal{M}$.) If an arbitrary Z has been given (as we shall often suppose) as π and $\bar\pi$ vary, $\partial_a Z$ will determine a 'crinkly cone' in $T^*\mathcal{M}$ that will not be quadratic and will give rise to a conformal Lorentzian analogue of a Finslerian structure.

At a fixed value of π and $\bar\pi$ one can obtain an expression for the metric that best approximates the crinkly cone at that value of π and $\bar\pi$ in terms of derivatives of Z and an arbitrary conformal factor by observing that if g^{ab} were independent of $\pi_{A'}$ and $\bar\pi_A$, then $g^{ab}\partial_a Z\,\partial_b Z = 0$ would imply

$$g^{cd}\partial^A\partial^B\partial^{A'}\partial^{B'}(\partial_c Z\,\partial_d Z) = 0 \tag{1}$$

This is 9 equations on 10 unknowns and therefore determines a 1-dimensional ray in the space of metrics, that is to say a conformal structure. This ray will in general vary with π and $\bar\pi$. The more explicit formulae obtained by Kozameh & Newman can usefully be written in this notation as well. This is a convenient framework for explicit calculations. However, it is complicated, and is not essential for an understanding of the programme.

We will also be interested in the 'non-Finslerian' condition—if a general choice of Z is made, the conformal structure determined by equation (1) will depend on $\pi_{A'}$ and $\bar\pi_A$. By taking an additional ∂^A derivative of (1) we see that the conformal structure will be independent of $\bar\pi_A$ iff $g^{ab}(\partial^{A'}\partial^{B'}\bar\eth^3(\partial_a Z\,\partial_b Z)) = 0$. The complex conjugate equation can be imposed to ensure independence from $\pi_{A'}$ also, although if everything is global in $\pi_{A'}$, g^{ij} will be global and holomorphic in $\pi_{A'}$ and therefore constant by Liouville's theorem. It will be shown in the next section that the weaker necessary scalar condition

$$g^{ab}\eth^3\bar\eth^3(\partial_a Z\,\partial_b Z) = 0 \tag{2}$$

is sufficient in linearized theory as this condition implies that g^{ab} is harmonic on the sphere so that the maximum principle holds—the only solutions should be conformal structures that are constant on the $(\pi_{A'},\bar\pi_A)$ sphere. This is probably also sufficient in the full nonlinear regime also. The condition that actually arises from the main field equation is different so this condition will not be studied any further.

§3 The main field equation. It is difficult to impose the vacuum field equations directly as the Ricci tensor depends on Ω as well as Z. The strategy I shall follow is to impose the Bach equations, $B_{ab}=0$ instead. These equations are conformally invariant and so constrain only Z. They are obtained from the conformally invariant Lagrangian $\int C^a{}_b \wedge {}^*C^b{}_a$ where $C^a{}_b = C^a{}_{bcd}dx^c \wedge dx^d$ is the Weyl tensor. The Bach tensor $B_{ab}=\nabla^c\nabla^d C_{acbd} - \frac{1}{2}R^{cd}C_{acbd}$ is symmetric, trace free, divergence free and vanishes when the Ricci tensor does. The Bach equations are a fourth order set of hyperbolic equations that lead to a unique evolution of initial data. If the initial data is constructed from a vacuum evolution of vacuum initial data, then the uniqueness implies that the vacuum evolution must agree with the Bach tensor evolution. Thus, if the Bach equations are imposed, the restriction to the vacuum equations can be implemented by choice of boundary conditions or initial data. In particular, if the space-time is obtained from Bach tensor evolution of the standard data at null infinity it must necessarily be conformal to vacuum. The main field equation will be the scalar equation

$$l^a l^b B_{ab}(x,\pi_{A'},\bar{\pi}_A)=0 \tag{3}$$

where $B_{ab}(x,\pi,\bar{\pi})$ is the Bach tensor as computed from the conformal structure determined by equation (1) at a fixed value of $(\pi,\bar{\pi})$, and $l_a=\partial_a Z(x,\pi,\bar{\pi})$. Clearly, as $\pi_{A'}$ and $\bar{\pi}_A$ vary we will obtain all the Bach equations when the conformal structure is independent of π and $\bar{\pi}$.

In linearized and half conformally flat theory this equation has the following remarkable features. The $l^a\partial_a$ derivatives can be integrated directly to yield an equation for Z as a function of π and $\bar{\pi}$ with the x coordinates merely parametrizing the solution space. The equation descends to an equation in $\mathcal{N}$ that, together with global considerations, determines which two spheres correspond to light cones of points of $\mathcal{M}$. Secondly, the solutions for Z, if global with the correct boundary conditions, lead to a conformal structure that is independent of π and $\bar{\pi}$ and hence satisfying the Bach equations.

This can be demonstrated in linearized theory as follows. In linearized theory $Z=x^{AA'}\pi_{A'}\bar{\pi}_A + z(x,\pi,\bar{\pi})$, $g^{ab}=\eta^{ab}+h^{ab}$ with z and h small and equation (1) reduces to:

$$h^{ab}=\partial^{A'}\partial^{B'}\partial^{A}\partial^{B}(Lz)$$

where $L = \pi^{C'}\bar{\pi}^{C}\partial_{CC'}$. Then a short computation shows that the main equation, (3), is just $L^5\eth^2\bar{\eth}^2 z = 0$. If the boundary conditions appropriate to the linearized vacuum equations are imposed, this can be integrated to give

$$\eth^2\bar{\eth}^2 z = \eth^2\bar{\sigma} + \bar{\eth}^2\sigma \tag{4}$$

where $\sigma = \sigma(x^{AA'}\bar{\pi}_A\pi_{A'}, \pi_{A'}, \bar{\pi}_A)$ etc. is the linearized asymtotic shear which is the free data for the field. This can be integrated directly by means of a Greens function to yield:

$$z(x,\pi,\bar{\pi}) = \oint |\pi_{A'}\pi'^{A'}|^2 \log|\pi_{A'}\pi'^{A'}|^2\left(\eth^2\bar{\sigma} + \bar{\eth}^2\sigma\right)\pi'_{A'}d\pi'^{A'}{}_\wedge\bar{\pi}'_A d\bar{\pi}'^A.$$

In order for this to yield an actual solution of the linearized vacuum equations we must have that h_{ab} as derived from the linearization of (1) is independent of π and $\bar{\pi}$. One can prove this in various ways, but the most suitable way for generalization is as follows.

Start with the main equation $L^5\eth^2\bar{\eth}^2 z = 0$. Act on this with the operator $N = \partial_{AA'}\partial^A\partial^{A'}$. It turns out that $[N, L^5] = \frac{5}{2}\Box L^4(\Upsilon + \bar{\Upsilon} + 6)$ where $\Upsilon = \pi_{A'}\partial^{A'}$ is the homogeneity operator and so the commutator vanishes on quantities of weight $(-3,-3)$ such as $\eth^2\bar{\eth}^2 z$. We also have the relation $N\eth^2\bar{\eth}^2 = \eth^3\bar{\eth}^3 L$ so that the main equation implies

$$\begin{aligned} 0 &= NL^5\eth^2\bar{\eth}^2 z = L^5 N\eth^2\bar{\eth}^2 z \\ &= L^5\eth^3\bar{\eth}^3(Lz). \end{aligned}$$

This integrates to yield $\eth^3\bar{\eth}^3(Lz) = 0$. However, $\eth\bar{\eth}h_{ab} = \pi_{A'}\pi_{B'}\bar{\pi}_A\bar{\pi}_B\eth^3\bar{\eth}^3(Lz)$ so this equation will imply that h_{ab} is harmonic on the $(\pi,\bar{\pi})$-sphere and so, by globality and the maximum principle it is independent of π and $\bar{\pi}$.

In the curved case, then, one expects that one can find a global solution $Z(x,\pi,\bar{\pi})$ to equation (3) given appropriate asymptotic data (to see existence, one needs only to use the asymptotic data to produce the space-time, and then use the space-time to produce Z). As a heuristic argument that this solution of equation (3) is unique with given boundary data, identify $(x,\pi,\bar{\pi})$ space with the total space of the spin bundle with coordinates $(x, \zeta_{\tilde{A}'}, \bar{\zeta}_{\tilde{A}})$ in the natural way (the tilde over the spinor indices for the spin bundle is to emphasize that they are not the same as those introduced above and furthermore that they should be considered to be abstract). The operator N will be represented as $\nabla_{\tilde{A}\tilde{A}'}\partial^{\tilde{A}'}\partial^{\tilde{A}}$ where $\nabla_{\tilde{A}\tilde{A}'}$ is the horizontal lift of the space-time derivative, and $\partial^{\tilde{A}} = \partial/\partial\zeta^{\tilde{A}}$ etc. If we act on the main equation $l^a l^b B_{ab}(x, \pi_{A'}, \bar{\pi}_A) = 0$ with N we obtain a scalar equation that is a necessary condition for the conformal structure to be non-Finslerian, since if the conformal structure is non-Finslerian, $Nl^al^bB_{ab}(x,\pi_{A'},\bar{\pi}_A)=0$ as a consequence of $\nabla^aB_{ab}=0$. We have seen above in the linearized case, with the appropriate boundary conditions, that this condition is also sufficient. It remains to prove that this condition will be sufficient in the curved case also. If so, then the space-time must necessarily satisfy $B_{ab}=0$ and have the given asymptotic data and therefore by uniqueness of evolution of the Bach equations be unique. Note that the equation $Nl^al^bB_{ab}(x,\pi_{A'},\bar{\pi}_A)=0$ on its own can be regarded as a kinematic equation that together with globality should lead to a non-Finslerian conformal structure—it is an identity, following from $\nabla^aB_{ab}=0$, for a genuine conformal structure.

§4 **The Yang-Mills case.** So far the explicit computation of equation (3) in terms of Z and its derivatives has proved intractable. However, the Yang-Mills analogue of the above ideas can be fully worked out in Minkowski space. Let $D_a=\partial_a-\gamma_a$ be a Yang-Mills connection on $\mathbf{M}$. The basic object of interest is the parallel propagator along light rays, a matrix valued funtion on the spin bundle of $\mathbf{M}$ $G(x,\pi,\bar{\pi})$ satisfying $\pi^{A'}\bar{\pi}^AD_{AA'}G(x,\pi,\bar{\pi})=0$. The connection is determined by the equation

$$\gamma^{AA'}=\eth^{A'}\bar{\eth}^A(LG\circ G^{-1}) \tag{1'}$$

where $L=\pi^{A'}\bar{\pi}^A\partial_{AA'}$ as above. If G has been chosen arbitrarily, γ_a will depend on π and $\bar{\pi}$ as well as x. The main field equation is

$$\pi^{A'}\bar{\pi}^AD^bF_{ab}=0 \tag{2'}$$

in which F_{ab} is the curvature of the connection γ_a. This yields the following equation on G:

$$L^3\bar{\eth}J+[\bar{J},L^3J]+3[L\bar{J},L^2J]=0. \tag{3'}$$

where $J=G^{-1}\eth G$. In the Abelian case this can be integrated to give $\eth\bar{\eth}$ log $G=\bar{\eth}A+\eth\bar{A}$ where $A=A(x^{AA'}\pi_{A'}\bar{\pi}_A,\pi_{A'},\bar{\pi}_A)$ is part of the asymptotic connection.

The analogue of the non-Finslerian condition can be stated as $\eth\gamma_a=\bar{\eth}\gamma_a=0$. As before, if we have globality, this will be guaranteed by the weaker condition $\eth\bar{\eth}\gamma_{AA'}=\pi_{A'}\bar{\pi}_A\eth^2\bar{\eth}^2(LG\circ G^{-1})=0$ using the maximum principle for solutions of the Laplacian on S^2. The non-Finslerian condition that arises from equation (2′) is obtained by acting on (2′) with $D_{AA'}\eth^{A'}\bar{\eth}^A$. Note that this vanishes automatically when γ_a is independent of π and $\bar{\pi}$ as a consequence of the Bianchi identity $D^aD^bF_{ab}=0$ so the vanishing of this quantity is a necessary condition if γ_a is to be non-Finslerian. In the linearized case it yields $L^3\eth^2\bar{\eth}^2(LG \circ G^{-1}) = 0$ which with suitable boundary conditions can be integrated to give the weaker version of the non-Finslerian condition. In the nonlinear case a more complicated equation is obtained, and its sufficiencey as a non-Finslerian condition has not yet been proved. Note that this equation if imposed on its own only constrains the γ_a dependence on π and $\bar{\pi}$, and there is no restriction on the x dependence.

§ 5 Conclusions and outlook: It seems likely that the main scalar equations, (3)and (3′) on Z and G are sufficient to encode the full Bach/Einstein equations and Yang-Mills equations. However, these equations are in effect articulated on the spin bundle of space-time, whereas the all important feature of the linearized and half flat cases is that the main field equation can be integrated directly to yield equations for G or Z on $\mathcal{N}$. Otherwise put, the equations of the linearized and half flat case yield precisely the condition that certain structures descend to $\mathcal{N}$ and that these structures can be used to write down the equations that determine Z and G. (In the half flat case the relevant structure is a complex structure.) It is this feature that leads to the twistor constructions. Unfortunately it is not even plausible that the main equations above will descend to $\mathcal{N}$ in this way, as it is easy to persuade oneself that if this were indeed possible, then the full Yang-Mills and Einstein equations would satisfy the Huygens property—that is, the solution at a point would depend only on the initial data at the intersection of the light cone at that point with the data surface. Instead, if the above ideas are to lead to a twistor construction for the full Yang-Mills and Einstein equation, there must be a further non-local transform to encode the structures on $\mathcal{N}$ or twistor space.

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Appendix : see p. 14