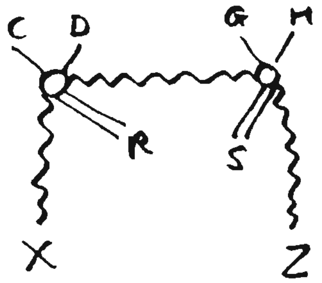
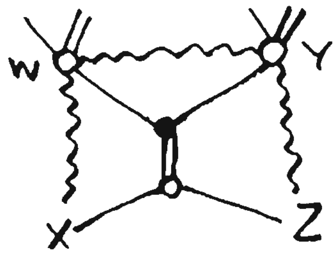
New contours with boundary, higher order diagrams, regularisation and massive propagators.

This note will further discuss and extend the "new" contours with boundary described in **TN30**. It turns out that these give the technical basis for a whole slew of advances in twistor diagram theory.

(1) An application of the spinor integral described in **TN30**. It follows from (2) on page 34 there, that there's a contour for the (projective) diagram

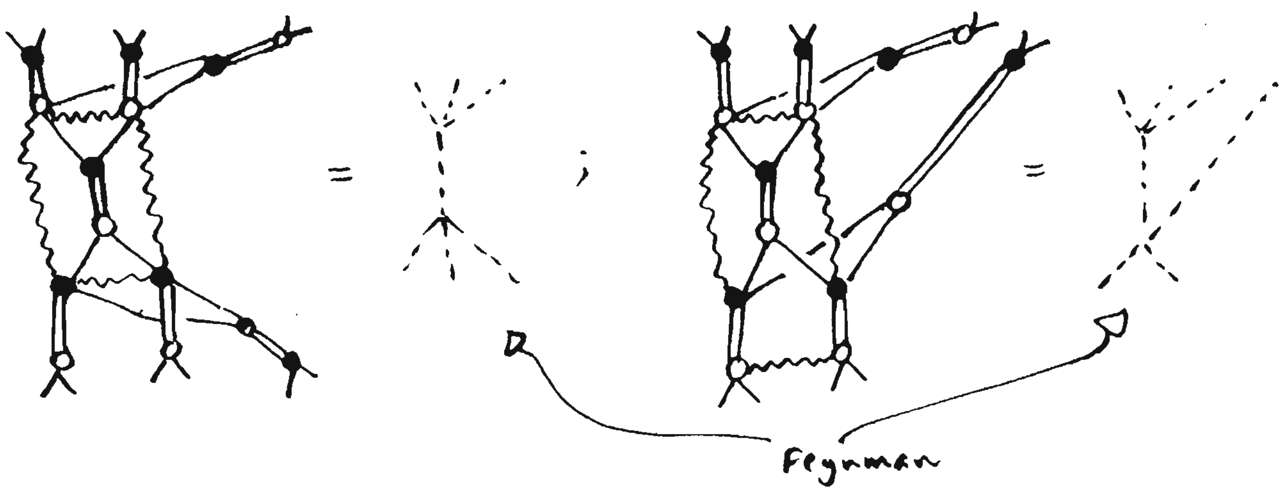


which allows CD = GH, R = S. This can be shown to induce a contour for



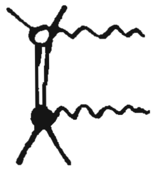
showing that the extra W. Z boundary line, as described in the article by L. J. O'D. in **TN 31**, is redundant.

(2) By an extension of this idea we can also produce contours for

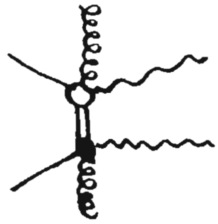


which makes some progress towards building higher order diagrams systematically.

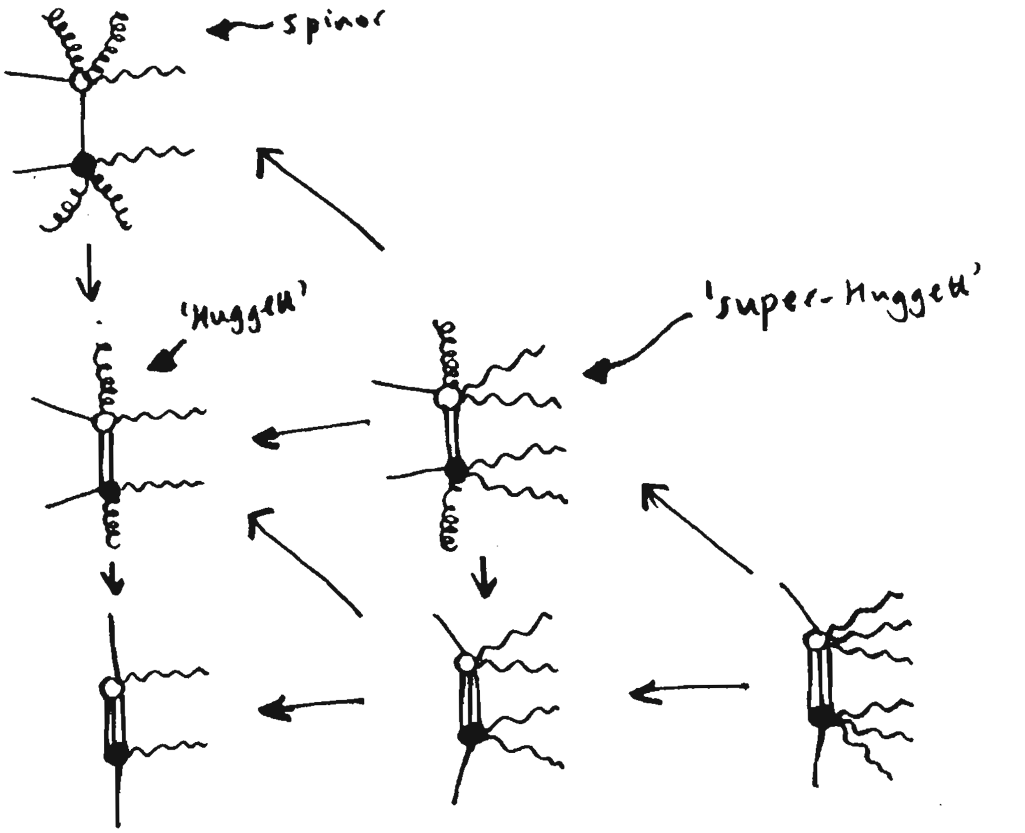
(3) In **TN30** I discussed explicit construction of the "Huggett" contour for the (projective) diagram



This can be thought of as



where the "springy" pole restricts from CP^3 to CP^2, so that the integral becomes a CP^2 analogy of the spinor integral just discussed. (The non-springy pole has the property of being inessential to the contour). Now this turns out to be just part of the following scheme of related integrals:



with relations:

↓ is just increase in dimension

← is "period"

↖ is through $\frac{\partial}{\partial Z^{\alpha}}\ Z \sim\!\sim\!\sim W = W_{\alpha}\ Z \sim\!\sim\!\sim W$

Contours will also exist when the internal line is not a pole but a boundary.

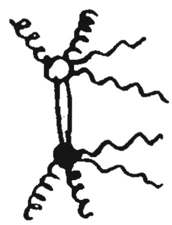
These cases are all of interest but the "super-Huggett" in CP^2 has a particular application. First I give an evaluation of it:

$$\oint\limits_{\substack{\xi\cdot\alpha_1=0,\ \xi\cdot\alpha_2=0\\ \eta\cdot\gamma_1=0,\ \eta\cdot\gamma_2=0}} \frac{D^2\xi\wedge D^2\eta}{(\xi\cdot\beta)(\eta\cdot\delta)(\xi\cdot\eta)^2} = \left\{ \begin{array}{l} \displaystyle\int_0^1 du\int_0^1 dv \frac{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \vdash & \dashv\\ \gamma_1 & \gamma_2\end{smallmatrix}\ \begin{smallmatrix}\alpha_1\\ |\\ \gamma_1\end{smallmatrix}}{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \vdash & \dashv\\ \gamma_1 & \delta\end{smallmatrix}\ \begin{smallmatrix}\beta & \alpha_1\\ \vdash & \dashv\\ \gamma_1 & \gamma_2\end{smallmatrix} - u\ \begin{smallmatrix}\alpha_1 & \alpha_2\\ \vdash & \dashv\\ \gamma_1 & \gamma_2\end{smallmatrix}\ \begin{smallmatrix}\alpha_1 & \beta\\ \vdash & \dashv\\ \delta & \gamma_1\end{smallmatrix} - uv\ \begin{smallmatrix}\beta\\ |\\ \delta\end{smallmatrix}\ \begin{smallmatrix}\alpha_1\\ |\\ \gamma_1\end{smallmatrix}\ \begin{smallmatrix}\alpha_1 & \alpha_2\\ \vdash & \dashv\\ \gamma_1 & \gamma_2\end{smallmatrix}} \\[2ex] + \text{ 3 similar terms obtained by} \\ \qquad \alpha_1 \leftrightarrow \alpha_2,\quad \gamma_1 \leftrightarrow \gamma_2 \end{array}\right.$$

which can if desired be expressed as the sum of 8 dilogarithms. By thinking of C^2 as (CP^2 - line at infinity), we can apply this integral to

$$\oint\limits_{\substack{x\cdot a=k_1,\ x\cdot b=k_2\\ y\cdot c=k_3,\ y\cdot d=k_4}} \frac{d^2x\wedge d^2y}{(x\cdot y-k)^2}$$

and so to the inhomogeneous twistor diagrams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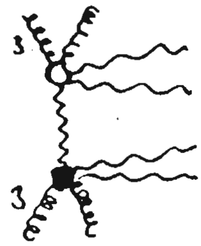
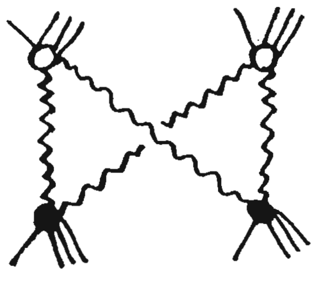
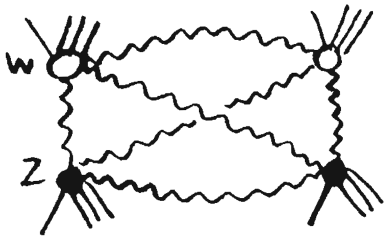


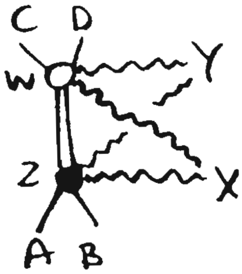
The point of this is that it allows a solution of the problem I described in **TN28**, that of finding a genuine contour with boundary for the inhomogeneous diagram



In that article I showed how the right answer for this diagram (which should be thought of as the *most fundamental* box diagram) could be obtained by a Pochhammer-contour construction involving the inhomogeneity in an essential way. This method *cheated*, however, in that I had to substitute logarithms for two of the boundary lines. I conjectured that this should not actually be necessary. The existence of the super-Huggett contour shows how we can avoid this cheat. Instead, what we need to do is to add in two extra inhomogeneous boundaries at infinity, i.e. consider



This diagram can be explicitly integrated to yield the right answer. The idea is to integrate out W_α and Z^α first. In fact I'll only give the result for the rather simpler case



$$\text{i.e.}\quad \oint_{\substack{W.X=k\\ Y.Z=k\\ \underset{\smile}{WY}=m,\ \overset{\frown}{ZX}=m}} \frac{D^4W\wedge D^4Z}{\underset{C}{\overset{W}{|}}\ \underset{D}{\overset{W}{|}}\ \underset{Z}{\overset{A}{|}}\ \underset{Z}{\overset{B}{|}}\ \Big(\underset{Z}{\overset{Y}{|}}-k\Big)^2}$$

because this has got just the same singularity structure in X^α and Y_α, while avoiding the complication due to the spin-1 fields. The result (obtained by applying the CP^2 integral formula given above) can be put in the form:

$$\frac{\pi^2}{6}-\operatorname{dilog}\left(\frac{-k\,\overset{AB}{\underset{CD}{\vdash\!\dashv}}}{\overset{YAB}{\underset{XCD}{\vdash\!\!\dashv}}}\right)-\operatorname{dilog}\left(\frac{-m^2\,\overset{AB}{\underset{CD}{\vdash\!\dashv}}}{k\,\begin{smallmatrix}X\\Y\end{smallmatrix}\overset{AB}{\underset{CD}{\vdash\!\dashv}}}\right)$$

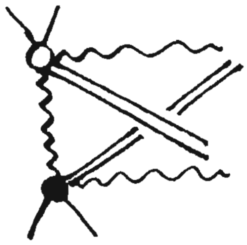
$$-\log\left(\frac{-k\,\overset{AB}{\underset{CD}{\vdash\!\dashv}}}{\overset{YAB}{\underset{XCD}{\vdash\!\!\dashv}}}\right)\log\left(\frac{-\frac{m^2}{k}\,\overset{AB}{\underset{CD}{\vdash\!\dashv}}}{\begin{smallmatrix}X\\Y\end{smallmatrix}\overset{AB}{\underset{CD}{\vdash\!\dashv}}}\right)$$

To proceed with the integration, we take a loop from Y.X = k, round the branch point at

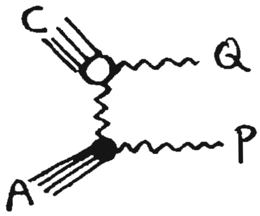
$$\overset{YAB}{\underset{XCD}{\vdash\!\!\dashv}} = -k\,\overset{AB}{\underset{CD}{\vdash\!\dashv}}$$

and back again. All terms but the first dilogarithm contribute zero and the computation thereafter runs just as in the "cheating" method. Since the result is conformally invariant, it seems very likely that the added boundaries on $\underset{\smile}{WY}$ = m, $\overset{\frown}{XZ}$ = m are in fact redundant and could be filled in by suitable "caps" to construct a contour with boundaries only as originally given. (However, the use of these extra boundaries is legitimate anyway according to current diagram philosophy, so this argument is not crucial).

Note: the article by L. J. O'D. in this **TN** describes the analogous construction in the rather simpler case of



(4) There is yet a further generalisation of these integrals. Note that the projective diagram



allows A. C = 0 (in fact the answer is then $\left(\frac{P.Q}{P.C\ A.Q}\right)^4$).

It follows that there is (in general position) a contour for the integral

$$\oint_{\substack{W.Z=0 \\ W.Q=0,\ P.Z=0}} \frac{DWZ}{(W.C)^3 (A.Z)^3 \begin{smallmatrix} W & A \\ Z & C \end{smallmatrix}}$$

For when A. C = 0 this is just the same integral; now move the parameters and the contour must survive. Transferring this observation to the inhomogeneous context, this tells us that there are contours for integrals of form

$$\oint_{\substack{W.Z=k \\ \cdots}} \frac{D^4WZ}{(W.Z)} \cdots$$

which we have already noted are crucial for regularisation (see the end of my article in **TN 31**, and also L. J. O'D in this **TN**.) It *also* tells us there are contours for integrals of form

$$\oint_{\substack{\overset{\frown}{XZ}=m \\ \cdots}} \frac{D^4X\, D^4Z}{(\overset{\frown}{XZ})} \cdots$$

and this now turns out to yield a key new idea in the incorporation of *massive* fields into twistor diagram theory.

To see this, note that the heart of the problem is the representation of the Feynman propagator function, which (in the scalar case) satisfies

$$(\Box_x + m^2)\Delta_F(x-y;m) = \delta(x-y)$$

A formal solution of this equation is given by

$$\Delta_F(x-y;m) = \sum_1^\infty (m^2)^{n-1}\,\Box^{-n}\delta(x-y)$$

but of course the inverse derivatives are so far undefined.
To proceed I shall compose everything with test fields, i.e. perform

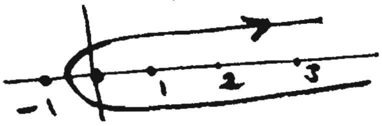
$$\iint d^4x\, d^4y\, ((x-r)^2)^{-2}((y-s)^2)^{-2}\ldots$$

We have

$$\iint d^4x\, d^4y\, \frac{\delta(x-y)}{((x-r)^2)^2((y-s)^2)^2} = \frac{1}{((r-s)^2)^2}$$

and

$$\iint d^4x\, d^4y\, \frac{\Delta_F(x-y;m)}{((x-r)^2)^2((y-s)^2)^2} = \int \frac{(m^2(r-s)^2)^{s-1}\,\Gamma(-s)\Gamma(1-s)\,ds}{\sin\pi s}$$



s-plane

This Barnes integral representation supplies a definitive specification of how the inverse derivatives are to be chosen: namely that the nth inverse derivative must correspond to the residue of this Barnes integral at $s = n-1$.

For instance it tells us that the $n = 1$ term is
$$\frac{\log(m^2(r-s)^2)+2\gamma}{(r-s)^2}$$

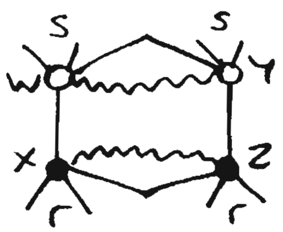
and the $n = 2$ term is
$$\frac{1}{2}\left[\log(m^2(r-s)^2)+2\gamma\right]^2 - \left[\log(m^2(r-s)^2+2\gamma-1\right] + \frac{\pi^2}{3}$$

These expressions can be thought of as regularisations of the divergent Fourier integrals

$$\int_{\substack{\text{timelike} \\ p^a}} d^4p \; \frac{e^{-ip.(r-s)}}{(p^2)^n}$$

We have earlier shown how infra-red and ultra-violet divergences can be regularised by inhomogeneous twistor diagrams. Can the same be done for these? Yes, it seems! The key idea is the combination of pole $(\overbrace{x\,z})^{-1}$ and boundary at $\{|\overbrace{x\,z}| = m\}$

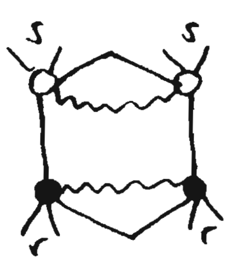
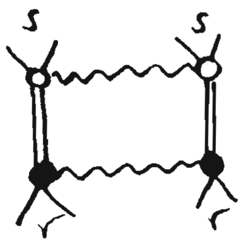
To see this explicitly, consider the diagram



where for temporary purposes I have indicated the factors $(\overbrace{x\,z})^{-1}$, $(\underbrace{w\,y})^{-1}$

by $\underbrace{x \quad z}$, $\overbrace{w \quad y}$

This formally satisfies

$\Box_r$  $=$ 

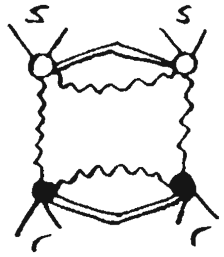
so if a contour exists such that the RHS integral yields $\left((r-s)^2\right)^{-2}$

then the LHS must correspond to a solution of $\Box_r^{-1}\left((r-s)^2\right)^{-2}$

In fact, such a contour does exist, and explicit evaluation gives

$$\frac{1}{(r-s)^2} \log\left(\frac{m^2}{k^2}(r-s)^2\right)$$

Similarly for n = 2, we can write down and then evaluate



$$= \frac{1}{2}\left[\log\left(\frac{m^2}{k^2}(r-s)^2\right)\right]^2 - \left[\log\left(\frac{m^2}{k^2}(r-s)^2\right) - 1\right] - \frac{\pi^2}{6}$$

(so that here the "new" contour with boundary plays an essential role.)

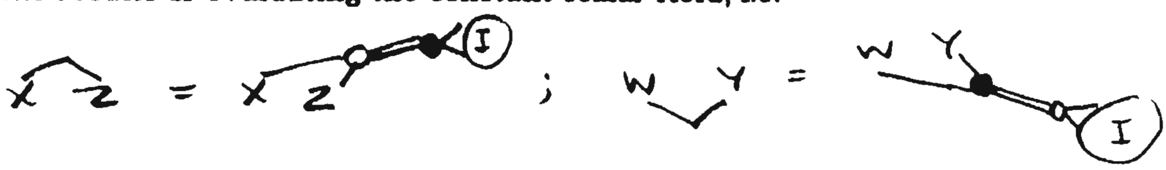
Obviously these results have general agreement in form with the expressions which sum to the Feynman propagator, but *exact* agreement implies something more. Firstly it suggests that we make a specific choice of the hitherto undetermined parameter k, namely

$$**** \qquad k = e^{-\gamma} \qquad ****$$

In fact this is essential if the m that enters into the specification of the boundary contour is to be equal to the m that appears in the coefficients of the power series.

Secondly, there remains the discrepancy of $\pi^2/2$. Examination shows that this can be removed provided we amend the prescriptions so that boundaries can lie on $\overset{\frown}{xz} = \pm m$, $\underset{\smile}{wy} = \pm m$. As yet this is only an *ad hoc* fix; underlying it is the fact that there are some questions of sign and time-direction which are closely involved in the definition of the Feynman propagator, and which have not yet been clarified. Furthermore it is not yet proved that that correct quantities can be generated for n = 3, 4, ... However, it seems very likely that there exists an integral scheme based on this contour structure which does allow the Feynman propagator (and *a fortiori*, the on-shell propagator) to emerge as a series of well-defined finite terms.

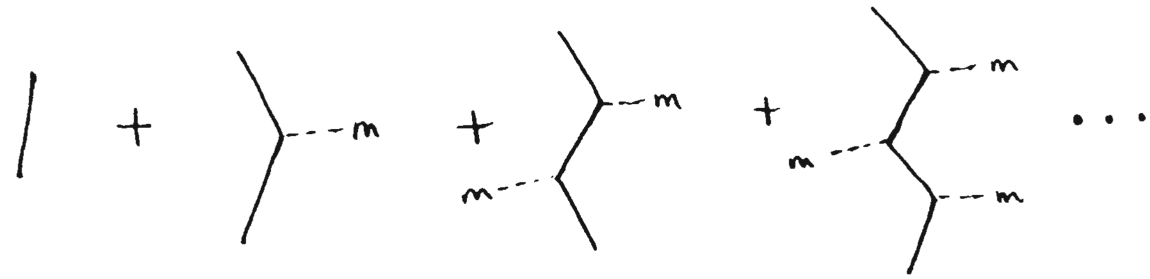
Note that the factors $(\overset{\frown}{xz})^{-1}$, $(\underset{\smile}{wy})^{-1}$, which play a key role here, are simply the results of evaluating the constant scalar field, i.e.



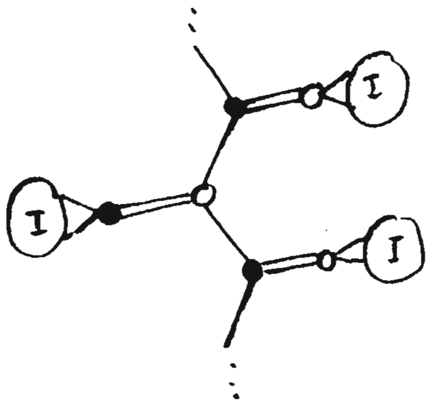
So the structure we are building here requires only the same ingredients as have been used in the description of massless field theory, with the sole addition of the constant field treated as an external field.

This suggests that the structure can be *directly* related to the "standard model" picture of massless fields acquiring mass through interaction with the constant Higgs field.

In the standard model the electron propagator is thought of as a sum over a series of (massless) Feynman diagrams of form



although this summation is purely formal, as none but the first of these has any finite meaning. But my prediction is that these divergent zigzag Feynman diagrams correspond systematically to perfectly finite twistor diagrams, each with a "skeleton" essentially of form



These should sum as an infinite series exactly to the correct massive propagator function. Much work still needed to show this though.

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