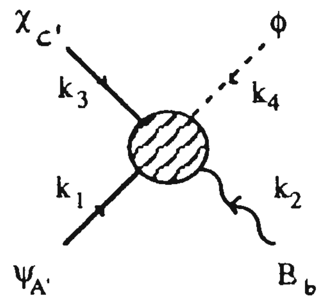
A twistor construction for gauge potentials on Minkowski space

We construct a twistor expression (6) satisfying twistor analogues of (1), (2) in order to facilitate calculations of scattering amplitudes with exterior gauge potentials.

As an example let us consider the lowest-order contribution to the amplitude of the following process arising from the Standard Model:



where $\psi_{A'}(k_1)$, $B_b(k_2)$ and $\bar{\chi}_C(-k_3)$, $\bar{\phi}(-k_4)$ are the (Fourier transforms of the) ingoing and outgoing (positive energy) fields, resp.. $B_b(k_2)$ is a U(1) gauge potential in Lorenz gauge

$$k_a B^a(k) = 0 \qquad (1)$$

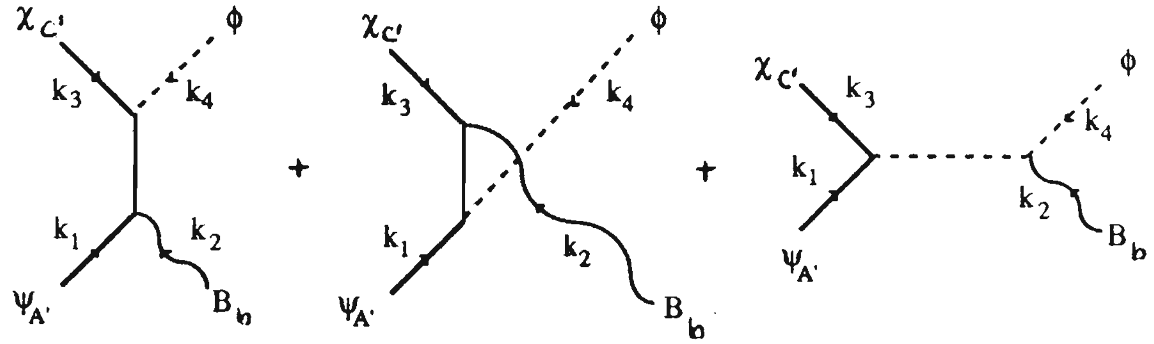
such that

$$F_{ab}(k) = \Theta_{(AB)}(k)\,\varepsilon_{A'B'} + \Psi_{(A'B')}(k)\,\varepsilon_{AB} = -ik_{AC'}\,B^{C'}{}_{B}(k)\,\varepsilon_{A'B'} - ik_{A'C}\,B^{C}{}_{B'}(k)\,\varepsilon_{AB} \qquad (2)$$

(in momentum space), where $\Theta_{(AB)}$, $\Psi_{(A'B')}$ satisfy massless free-field equations.

Summing the contributing Feynman diagrams which, taken separately, are not gauge invariant

(I) (II) (III)



we get a momentum space integral (massless theory):

$$\int d^4k_1\, d^4k_2\, d^4k_3\, d^4k_4\, \delta^+(k_1)\, \delta^+(k_2)\, \delta^-(k_3)\, \delta^-(k_4)\, \delta(k_1+k_2+k_3+k_4)\, (\mathrm{I+II+III}) \quad \text{with}$$

$$\mathrm{I + II + III} = \phi(k_4)\, \chi_{C'}(k_3)\, B_{BB'}(k_2)\, \psi_{A'}(k_1)\, \Delta^{A'BB'C'} \qquad (3)$$

up to some costant factors (coupling constants, i's, ...),

$$\text{where} \quad \Delta^{A'BB'C'} = \frac{(k_1+k_2)^{BC'}}{k_1k_2}\varepsilon^{A'B'} + 2\frac{k_4^{BA'}}{k_1k_4}\varepsilon^{B'C'} + \frac{k_4^{BB'}}{k_1k_3}\varepsilon^{A'C'}$$

'=' after ∫-tion

$$\cong \frac{-k_4^{BC'}}{k_1k_2}\varepsilon^{A'B'} + 2\frac{k_4^{BA'}}{k_1k_4}\varepsilon^{B'C'} + \frac{k_4^{BB'}}{k_1k_3}\varepsilon^{A'C'}$$

It then involves quite some spinor algebra, using

$$(k_1 + k_2 + k_3 + k_4) \cong 0\,, \quad k_{1A}{}^{A'}\psi_{A'}(k_1) = k_{3A}{}^{A'}\chi_{A'}(k_3) = 0 \quad , \tag{4}$$

$$k_{2CB'}B^{B'}{}_{B}(k_2) = i\Theta_{(CB)}(k_2)\,, \; k_{2C'B}B^{B}{}_{B'}(k_2) = i\Psi_{(C'B')}(k_2) \quad , \tag{5}$$

to show how the above integral only depends on $\Theta_{(CB)}$, $\Psi_{(C'B')}$ and not on the gauge (cf. [1]). In fact, one finds that the term involving $\Psi_{(C'B')}$ vanishes (see (9)) and therefore one has helicity conservation (i.e. total incoming helicity = 0).

Since the relations (4) are already implicit in the algebra of a twistor diagram one is led to consider a (formal) twistor expression for $B_a(k_2)$ which also entails (1), (2) ($\Leftrightarrow$ (5)). As one has the correspondence of operators

$i\partial_{XAA'}$ (Minkowski space) $\leftrightarrow k_{AA'}$ (momentum space) $\leftrightarrow -X\partial_X$ (on generating functions on twistor space)

we represent

$$B_a(k) \leftrightarrow X\,V(\partial_X V)^{-1} f_{-4}(X^\alpha) + U\partial_X(XU)^{-1} f_0(X^\alpha) \tag{6}$$

where V_α, U^α are auxiliary twistors (corresponding to some gauge freedom) and $f_{-4}(X^\alpha)$, $f_0(X^\alpha)$ are twistor functions of homogeneity -4 , 0 resp. . Then

$$k_a B^a(k) \leftrightarrow XX\partial_X V(\partial_X V)^{-1} f_{-4}(X^\alpha) + \partial_X\partial_X XU(XU)^{-1} f_0(X^\alpha) = 0 \tag{7}$$

and

$$\Theta_{(AB)} = -ik_{AC'}B^{C'}{}_{B} \leftrightarrow iXX\,\partial_X V(\partial_X V)^{-1} f_{-4}(X^\alpha) + XU\,\partial_X\partial_X(XU)^{-1} f_0(X^\alpha) = \partial_X\partial_X\, i f_0(X^\alpha)$$

$$\Psi_{(A'B')} \leftrightarrow XX\, i f_{-4}(X^\alpha) \tag{8}$$

independent of V_α, U^α. This construction facilitates our calculations quite a bit. For example it is straightforward to verify that (3) = 0 for $F_{ab} = \Psi_{(A'B')}\varepsilon_{AB}$:

Let

$$\psi_{A'}(k_1) \leftrightarrow Z f_{-3}(Z^\alpha)\,, \quad \chi_{C'}(k_3) \leftrightarrow \partial_W\, g_{-1}(W_\alpha)\,,$$

$$B_{BB'}(k_2) \leftrightarrow XV(\partial_X V)^{-1} h_{-4}(X^\alpha)\,, \quad \phi(k_4) \leftrightarrow j_{-2}(Y_\alpha)$$

be representations of the exterior fields and potential by (dual) twistor functions. Then

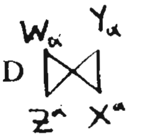
$$\psi_{A'}B_{BB'}\chi_{C'}\phi\,\Delta^{A'BB'C'} \leftrightarrow \Big\{ Z\,X\,V(\partial_X V)^{-1}\partial_W \Big[\sqcap \lfloor Y \lceil \partial_Y (Z\,X\,\partial_Z\partial_X)^{-1}$$

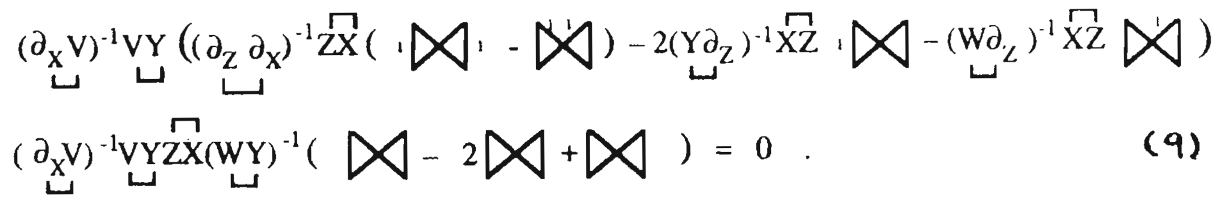
$$+ 2(-\lceil\partial_Y \lfloor Y)\sqcap(Z\,\partial_Y Y\partial_Z)^{-1} + \big[(-\lfloor Y \lceil\partial_Y\big](Z\,\partial_W W\partial_Z)^{-1} \Big] \Big\}_{contr} \quad F$$

with F = fghj. The twistors outside are contracted into the [ ] - bracket from left to right in the obvious way.

$$= (\partial_X V)^{-1} VY\,[ZX\,\partial_W\partial_Y(ZX\,\partial_Z\partial_X)^{-1} - 2\,Z\partial_Y\,X\partial_W(Z\partial_Y\,Y\partial_Z)^{-1} - Z\partial_W\,X\partial_Y(Z\partial_W\,W\partial_Z)^{-1}]F$$

$=: DF$.

Operating on the single box this yields D  =



(9)

The amplitude corresponding to the a.s.d. part of F_{ab} leads to a twistor diagram independent of U^{α} corresponding to the spinor expression

$$\Delta^{A'BB'C'}\phi(k_4)\chi_{C'}(k_3)B_{BB'}(k_2)\psi_{A'}(k_1) = \frac{k_1k_2 - k_1k_3}{(k_1k_2)(k_2k_3)(k_3k_1)} k_{1E'}{}^{B} k_3{}^{DE'} \chi^{A'}\psi_{A'}\Theta_{DB}\phi \ .$$

Similarly in the non-abelian case a perturbative expansion in powers of coupling constants (g^n) relates the exterior free fields $\overset{o}{\Psi}_{(A'B')\Theta}{}^{\Phi}$, $\overset{o}{\Theta}_{(AB)\Theta}{}^{\Phi}$ of order g^0 in

$$\overset{o}{F}_{ab\Theta}{}^{\Phi} + g\overset{1}{F}\ldots = \varepsilon_{AB}\overset{o}{\Psi}_{A'B'\Theta}{}^{\Phi} + \varepsilon_{A'B'}\overset{o}{\Theta}_{AB\Theta}{}^{\Phi} + g\varepsilon_{AB}\overset{1}{\Psi}\ldots$$

$$= 2\partial_{[a}A_{b]\Theta}{}^{\Phi} - 2igA_{X[a}{}^{\Phi}A_{b]\Theta}{}^{X} \qquad (10)$$

linearly to the gauge potentials (again taken to satisfy the Lorenz gauge condition $\nabla_a A^a{}_\Theta{}^\Phi = \partial_a A^a{}_\Theta{}^\Phi = 0$). An analogous construction as in (6) can therefore be used.

One can apply this also to cases of higher helicities, such as for example in [2], as long as one is over flat space. If one looks at these potentials classically one has to ask, however, what their space-time version looks like (i.e. how they are to be 'contour-integrated') and they might turn out not to be very general.

[1] F. Müller (1991). Twistor Diagrams for some scattering amplitudes arising from the Standard Model. Qualifying dissertation.

[2] R. Penrose (1990). Twistor Theory for Vacuum Space-Times: a New Approach, T N **31**, 6–8.

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Twistors as Spin 3/2 Charges

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Abstract It is pointed out that twistors play a role as the charges for helicity 3/2 massless fields. Since such fields can be defined consistently in general Ricci-flat 4-manifolds, a possible new approach to defining twistors in vacuum