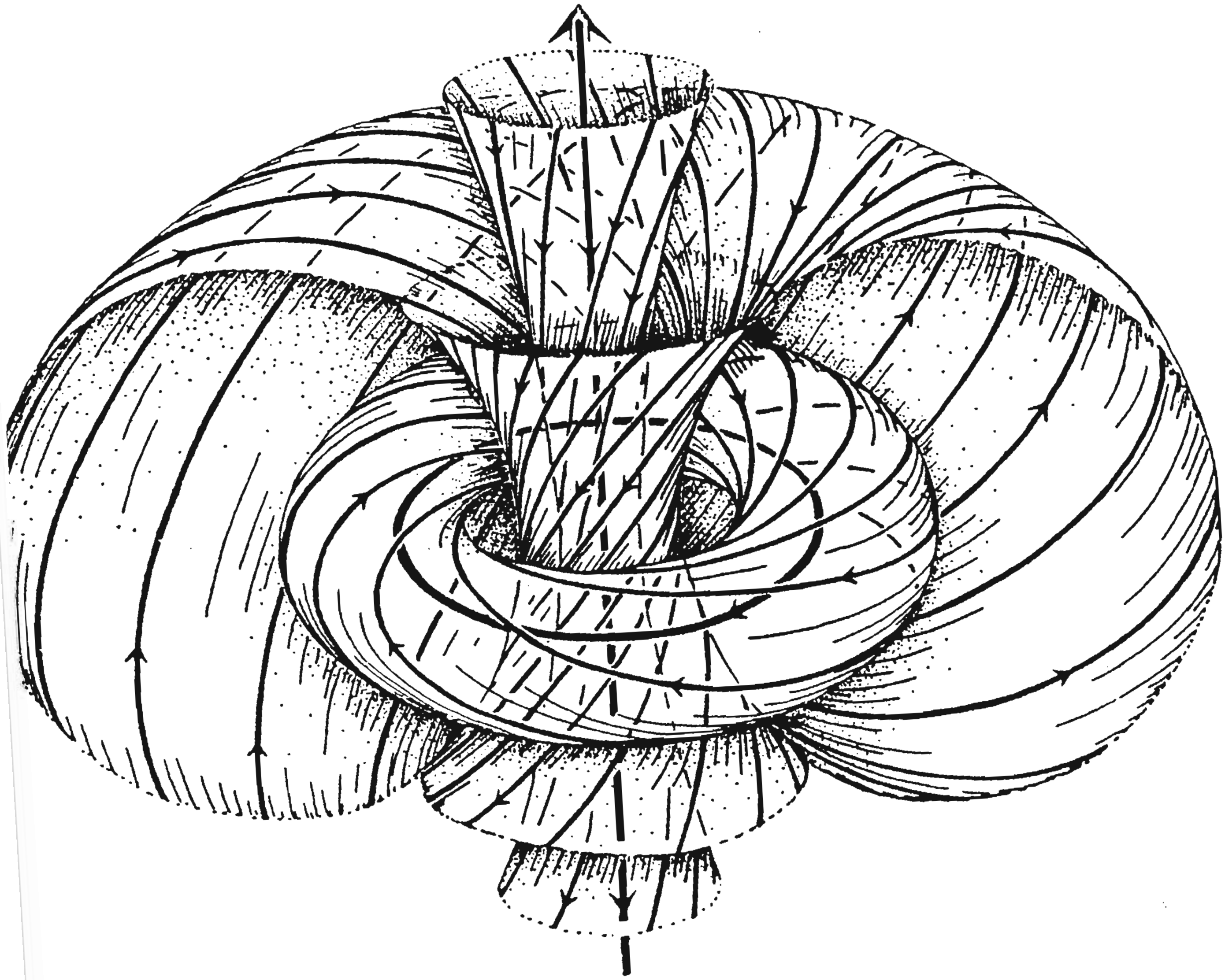
Twistor Newsletter (no 32: 12, March 1991)



Mathematical Institute Oxford, England

Twistors as Charges for Spin $\frac{3}{2}$ in Vacuum

In TN 31 (1990) pp. 6–8, I put forward the suggestion that it might be possible to develop a twistor theory for general vacuum space-times by describing a twistor as a kind of helicity $\frac{3}{2}$ <u>elemental state</u>. Elemental states were introduced by APH in TN 22 (1986), reprinted in F.A.T.T., vol 1 (1990). They provide a cohomological space-time interpretation of the internal lines $\circ\!\!-\!\!\bullet$ that make up twistor diagrams. For spin $\frac{3}{2}$, the elemental states would correspond to the internal lines "$\circ\overset{-2}{-}\bullet$" that occur with perturbative descriptions of gravitational scatterings, regarding "$W\overset{-2}{-}Z$" as a "twistor function" $f(W_\alpha)$ of homogeneity degree -1 describing a twistor Z^α as some kind of cohomological space-time field (of helicity $\frac{3}{2}$, since $f(W_\alpha)$ has homogeneity $1 = 2s-2 = 2\times\frac{3}{2}-2$). The idea here was to use the fact that spin $\frac{3}{2}$ fields are, in an appropriate sense, "consistent" in (and only in) Ricci-flat space-times (see H.A. Buchdahl, Nuovo Cim. 10 (1958) 96–103; S. Deser & B. Zumino, Phys. Lett. 62B (1976) 335–7; and especially B. Julia, Comptes Rendus (1982)), so the interpretation of twistors as helicity $\frac{3}{2}$ elemental states might also carry over to general Ricci-flat space-times. An essential problem had seemed to be how to characterize the particular degree 1 "twistor function $f(W_\alpha)$", in terms of its cohomological helicity $\frac{3}{2}$ space-time field that actually picks out $W\overset{-2}{-}Z$ amongst all twistor functions $f(W_\alpha)$. However, in flat space there is a special cohomological role for $W\overset{-2}{-}Z$, which can be reinterpreted in space-time terms, as providing Z^α as playing a role as a ("disembodied") <u>charge</u> for a helicity $\frac{3}{2}$ field.

There are various different formulations of the massless spin $\frac{3}{2}$ equations. The one I used in TN 3 might be called the "<u>Dirac form</u>" where, in flat space-time $\mathbb{M}$ we have the chain

$$\psi_{A'B'C'} = \psi_{(A'B'C')}\,, \quad \nabla^{AA'}\psi_{A'B'C'} = 0$$

$$\gamma^{A}_{B'C'} = \gamma^{A}_{(B'C')}\,, \quad \nabla^{BB'}\gamma^{A}_{B'C'} = 0\,, \quad \nabla_{AA'}\gamma^{A}_{B'C'} = \psi_{A'B'C'}$$

$$\rho^{AB}_{C'} = \rho^{(AB)}_{C'}\,, \quad \nabla^{CC'}\rho^{AB}_{C'} = 0\,, \quad \nabla_{BB'}\rho^{AB}_{C'} = \gamma^{A}_{B'C'}$$

with gauge freedom

$$\gamma^{A}_{B'C'} \longmapsto \gamma^{A}_{B'C'} + \nabla^{A}_{B'}\nu_{C'} \qquad \text{with} \quad \nabla^{AC'}\nu_{C'} = 0$$

$$\rho^{AB}_{C'} \longrightarrow \rho^{AB}_{C'} + \varepsilon^{AB}\nu_{C'} + i\,\nabla^{A}_{C'}\chi^{B} \quad \text{with} \quad \nabla_{AC'}\chi^{A} = 2i\,\nu_{C'}\,.$$

There is also a "gauge freedom of the second kind" (as in EM-theory) which leaves the potentials unchanged, given by

$$\nu_{A'} = \Pi_{A'}, \quad \chi^A = \Omega^A$$

where $(\Omega^A, \Pi_{A'})$ are the spinor parts of some twistor, i.e.

$$\nabla_{AA'}\Omega^B = -i\,\varepsilon_A{}^B\,\Pi_{A'} \qquad \text{with } \Pi_{A'} \text{ constant.}$$

We have an exact sequence

$$0 \to \underset{(\Omega,\Pi)}{\text{gauge patts of 2nd kind}} \to \underset{(\chi,\nu)}{\text{gauge quantities}} \to \underset{(\rho,\gamma)}{\text{potential}} \to \underset{(\psi)}{\text{field}} \to 0$$

In a general Ricci-flat curved space-time M, most of this does not carry through, but we can still retain (Dirac form)

$$\gamma^A_{B'C'} = \gamma^A_{(B'C')}, \quad \nabla^{BB'}\gamma^A_{B'C'} = 0, \quad \text{with gauge } \nabla^A_{B'}\nu_{C'} \text{ where } \nabla^{AC'}\nu_{C'} = 0$$

as a consistent system (subject to some puzzling phenomena that we shall need to come back to later). (It is easy to see that the Ricci-flat condition is necessary and sufficient for the gauge freedom to be actually $\nabla^A_{B'}\nu_{C'}$; that it is sufficient also for the γ equations is more complicated.) People who work with supersymmetry, however, prefer what one may call the "Rarita-Schwinger form" where we do not impose symmetry $\gamma^A_{B'C'} = \gamma^A_{(B'C')}$, and take our equations as

$$\varepsilon^{B'C'}\nabla_{A(A'}\gamma^A_{B')C'} = 0, \quad \nabla^{B'(B}\gamma^{A)}_{B'C'} = 0, \quad \text{with gauge } \nabla^A_{B'}\nu_{C'} \text{ where } \nu_{C'} \text{ is unrestricted.}$$

The first of these equations asserts the total symmetry of $\psi_{A'B'C'} = \nabla_{A(A'}\gamma^A_{B')C'}$; however, it should be noted that if M is not a.s.d. (anti-self-dual), then ψ is not gauge-invariant (in either the Dirac or R-S form). The R-S form is easily reduced to the Dirac form by using a gauge choice for ν (solving an inhomogeneous Weyl neutrino equation) which makes γ symmetric, the remaining gauge freedom being subject to $\nabla^{AA'}\nu_{A'} = 0$.

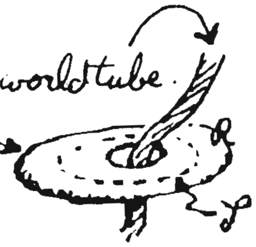
In $\mathbb{M}$, we may obtain the charges for a helicity $3/2$ field by spin-lowering. Take the spinor field $\mu^{A'}$ to be the primary part of a dual twistor $W_\alpha = (\lambda_A, \mu^{A'})$:

$$\nabla_{AA'}\mu^{B'} = i\,\varepsilon_{A'}{}^{B'}\lambda_A, \qquad \lambda_{A'} \text{ const.}$$

and set

$$\varphi_{A'B'} = \psi_{A'B'C'}\mu^{C'}$$

for some ψ in an open region $\mathcal{R}$ surrounding a worldtube.



We can use the s.d. Maxwell field φ in a charge integral over some 2-surface $\mathcal{S}$ in $\mathcal{R}$ surrounding the tube:

$$\frac{i}{4\pi}\oint \varphi_{A'B'}\, dx_A^{A'} \wedge dx^{AB'} = \zeta(W) \quad = \text{electric} + i \times \text{magnetic}$$

where $\zeta(W)$ is linear in W_α and therefore $\zeta(W) = Z^\alpha W_\alpha$ for some Z^α. This Z^α is the required charge for ψ. Note that it does not depend on the particular choice of $\mathcal{S}$.

We want to do something here that has a chance of holding in $\mathcal{M}$ and hence of providing a definition of a twistor in general Ricci-flat space-times (and independent of any particular choice of $\mathcal{S}$). A possible suggestion would be to generalize the following, which works in $\mathbb{M}$: Consider fields ψ... which are global in $\mathcal{R}$ ($\subset \mathbb{M}$). We shall say that two such fields are equivalent if their difference has a (second) potential ρ (and therefore also a potential γ) which is global in $\mathcal{R}$. The space $\mathbb{T}^\alpha$ of twistors can then be identified with this space of equivalence classes

$$\mathbb{T}^\alpha = \{\text{global } \psi\text{s}\} \,/\, \{\text{global } \rho\text{s}\} .$$

We can also find the space $\mathbb{S}_{A'}$ of constant $\pi_{A'}$-spinors, and the space $\mathbb{S}^A$ of constant ω^A-spinors when $\pi_{A'} = 0$ as, respectively:

$$\mathbb{S}_{A'} = \{\text{global } \psi\text{s}\}/\{\text{global } \gamma\text{s}\}, \qquad \mathbb{S}^A = \{\text{global } \gamma\text{s}\}/\{\text{global } \rho\text{s}\}$$

illustrating

$$0 \to \mathbb{S}^A \to \mathbb{T}^\alpha \to \mathbb{S}_{A'} \to 0 .$$

How much of this generalizes for a Ricci-flat $\mathcal{M}$? It may be remarked that if $\mathcal{M}$ is a.s.d. then ψ-fields exist and have the same relation to γs as in $\mathbb{M}$. This is consistent with the above and with the fact that $\mathbb{S}_{A'}$ is well-defined in the a.s.d. case. If $\mathcal{M}$ is s.d. then ρ-potentials exist (locally) and have the same relation to γs as in $\mathbb{M}$, consistent with the fact that $\mathbb{S}^A$ is well-defined in the s.d. case. In the a.s.d. case we can hope to go further and obtain the full curved twistor space $\mathcal{T}$ of the "non-linear graviton" construction. Although we do not have ρs locally, we can weaken the equation relating ρ to γ to

$$\gamma^A_{B'C'} = \nabla_{B(B'}\hat{\rho}^{AB}_{\ \ C')} ,$$

since this should be locally soluble, and then state the equivalence between two ψs in terms of the existence of a complex 2-surface (the α-surface of "Z") on which their respective $\hat{\rho}$s have an appropriate globality. This criterion seems too clumsy to be what is ultimately needed, but something of a non-linear nature is certainly required.

If $\mathcal{M}$ is general (Ricci-flat), or even merely s.d., we need a concept of a "global ψ" throughout $\mathcal{R}$ ($\subset \mathcal{M}$), even though the actual spinor ψ is not gauge invariant. In the case of $\mathbb{M}$, we can state such globality entirely in terms of γ by saying that $\mathcal{R}$ can be covered by open sets $\{\mathcal{U}_i\}$ where $\exists\, \overset{i}{\gamma}$ in $\mathcal{U}_i$ with

$$\overset{i}{\gamma} - \overset{j}{\gamma} = \nabla \overset{ij}{\nu} \quad \text{in each (non-empty) } \mathcal{U}_i \cap \mathcal{U}_j$$

where

$$\overset{ij}{\nu} \text{ exists in } \mathcal{U}_i \cap \mathcal{U}_j$$

with $\nabla^{AC'} \overset{ij}{\nu}_{C'} = 0$ (taking the Dirac form). We can then define

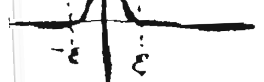
$$\overset{ijk}{\Pi} = \overset{ij}{\nu} + \overset{jk}{\nu} + \overset{ki}{\nu} \quad \text{in } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$$

and note that the Πs are constant since $\nabla \overset{ijk}{\Pi} = 0$. (We could also do the same with the ρs, χs and Ωs to obtain a twistor $(\overset{ijk}{\Omega}{}^A, \overset{ijk}{\Pi}_{A'})$ in $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$.) Appropriately adding these up, we obtain $(?\, \frac{1}{2\pi i} \times)\, \pi_{A'}$ (or $\frac{1}{2\pi i}(\omega^A, \pi_{A'})$) as required. This is an example of an "evaluation procedure" for a cohomology class — here an H^2 element, so we move 3 steps back from ψ in our exact sequence of potentials in order to get the charge for ψ.

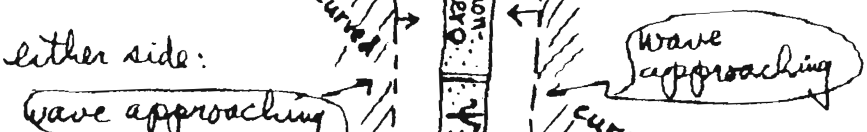
In a (non-a.s.d.) $\mathcal{M}$ something must go wrong with our original γs if we are to expect a non-zero $\pi_{A'}$-charge. For there are no constant $\Pi_{A'}$ spinors other than zero. This seems to be an almost paradoxical situation, because we can envisage situations where a "ψ" with a non-zero charge is set up in an initially flat region of space-time, and this "ψ" evolves, within its domain of dependence, into a full region of non-zero curvature. (Here "ψ" means γs modulo $\nabla\nu$s in a covering of $\mathcal{R}$, as described above.) We can arrange that the initial ψ-field is non-zero entirely within a thin slab, and singular only just around the centre of the slab, e.g. of the form

$$\zeta^{-1} o_{A'} o_{B'} o_{C'} B(u) \quad \text{or} \quad \bar{\zeta}^{-1} \iota_{A'} \iota_{B'} \iota_{C'} B(v)$$

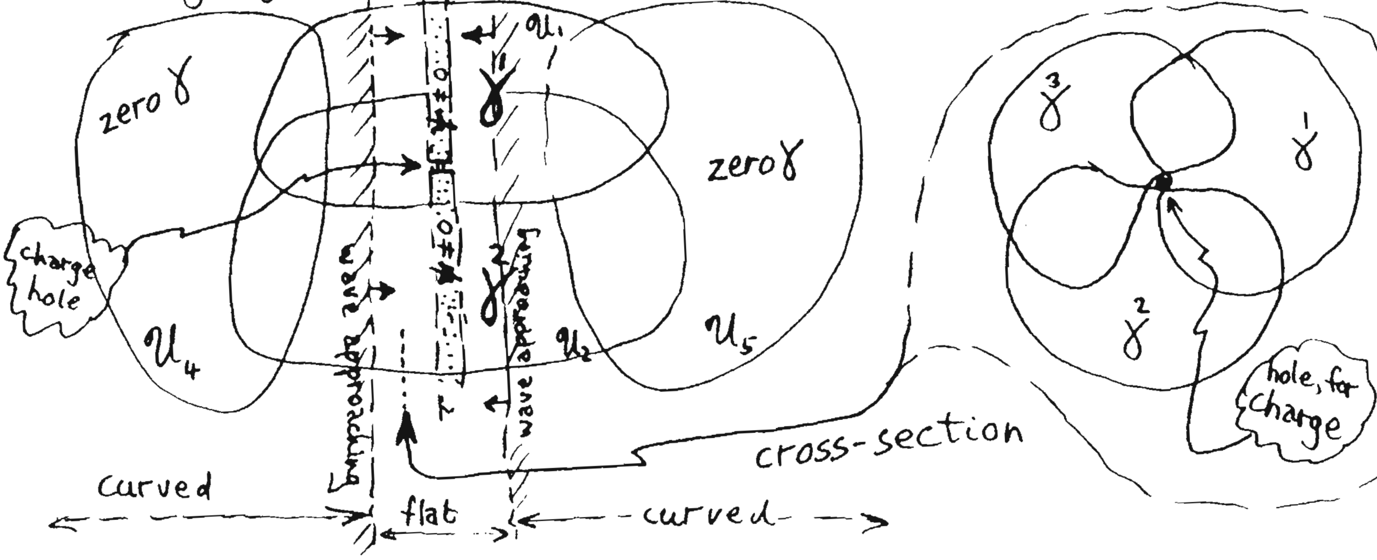
at $u+v=0$ ("$t=0$") in a flat region with standard null coordinates $(u, v, \zeta, \bar{\zeta})$ and constant dyad (o^A, ι^A) for $\mathbb{M}$ (metric $ds^2 = 2du\,dv - 2d\zeta d\bar{\zeta}$, $\frac{\partial}{\partial v} = o^A \bar{o}^{A'} = \ell^a$, $m^a = \frac{\partial}{\partial \zeta}$, $\frac{\partial}{\partial u} = \iota^A \bar{\iota}^{A'} = n^a$) where B is a C^∞ "bump function", non-zero only in $(-\varepsilon, \varepsilon)$. Plane waves approach this region from



either side:

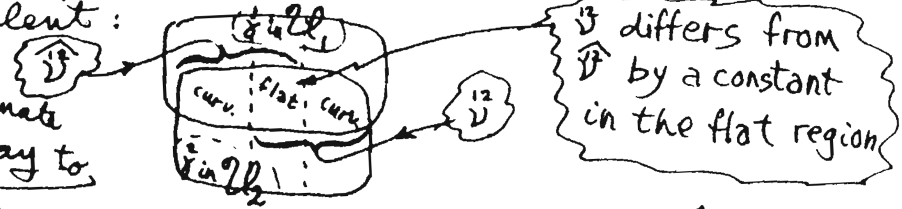


It would ~~seem~~ seem that we could cover a huge region of $\mathcal{M}$ with large open sets with large overlaps — large enough so that the domains of dependence of these open sets, individually, still overlap after the waves have collided and the space-time is

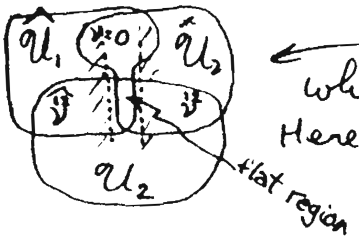


entirely curved. Plane waves have constant spinors but we can tilt them very slightly ~~so~~ that these constants are not in the directions of o^A or ι^A. Thus the $\pi_{A'}$-charges must disappear (these being in the directions of $o_{A'}$ or $\iota_{A'}$. How can this happen?

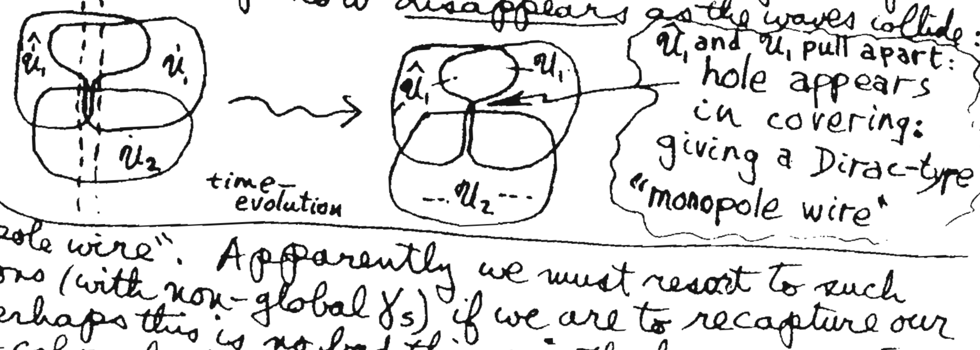
What actually occurs in this situation is somewhat surprising. It turns out that we find pairs of γs that are "locally" but not properly globally gauge equivalent:



To make this legitimate we must subdivide, say to,



where $\hat{\gamma}$ and $\overset{1}{\gamma}$ are actually identical in $\hat{\mathcal{U}}_1 \cap \mathcal{U}_1$! Here, all 3 of $\hat{\gamma}$, $\overset{1}{\gamma}$ and $\overset{2}{\gamma}$ are properly gauge equivalent on their overlaps. However, the needed region of triple overlap now disappears as the waves collide:



It seems that the twistor-charge "escapes" along a Dirac-type "monopole wire". Apparently we must resort to such "wire-type" descriptions (with non-global γs) if we are to recapture our twistor concept. Perhaps this is no bad thing in the long run, since a non-linear (non-cohomological, in the strict sense) concept is needed. Work is in progress. There are further relations to twistor diagrams that will be reported on later.

Especial thanks go to K.P.T., who worked out the actual propagation of the above charged fields and potentials into plane-wave backgrounds — showing that we do not get a Hans Lewy obstruction (at least at this stage) as had seemed possible; also to L.J.M. for putting me straight on the R-S. equations. Comments from A.P.H., R.J.B. and T.S.-T. have also been valuable. — Roger Penrose

On the Topology of Quaternionic Manifolds

Claude LeBrun

March 11, 1991

Recall that a Riemannian manifold (M,g) of dimension $4k$, $k > 2$, is said to be *quaternionic-Kähler* if its holonomy group is a subgroup of $[Sp(k) \times Sp(1)]/Z_2$. Such manifolds are interesting, not only because they are automatically Einstein and occur in the holonomy classification of Riemannian geometries, but also because they have twistor spaces analogous to those familiar from dimension 4. Indeed, in the special case of $k = 1$, we define (M,g) to be *quaternionic-Kähler* iff it is self-dual Einstein.

The only known examples of compact quaternionic-Kähler manifolds of scalar curvature $R \neq 0$ are symmetric spaces, and indeed there are theorems [2, 4] asserting that a compact quaternionic-Kähler manifold of dimension 4 or 8 with $R > 0$ *must* be symmetric. While it remains unclear whether or not such a result might hold in higher dimensions, I would like to point out that, in this direction, one can at least say the following:

Theorem 1 *Let (M,g) be a compact quaternionic-Kähler $4k$-manifold with $R > 0$. Then either*
(a) $b_2(M) = 0$; or else
(b) M is the symmetric space $G_2(C^{k+2}) = SU(k+2)/S(U(k) \times U(2))$.

With a little bit of twistor theory, this is in fact an immediate consequence of some recent advances in algebraic geometry stemming from Mori's theory of extremal rays [3]. Let us begin by recalling that a compact quaternionic-Kähler $4k$-manifold with $R > 0$ has as its twistor space a compact complex $(2k+1)$-manifold Z which admits a complex contact form and a Kähler-Einstein metric with positive scalar curvature. In particular, $c_1 > 0$, so that Z is a so-called *Fano manifold*, and c_1 is divisible by $k+1$. We can therefore invoke the following result of Wiśniewski [6]:

Theorem 2 *(Wiśniewski). Let X be a Fano manifold of dimension $2r-1$ for which $r|c_1$. Then $b_2(X) = 1$ unless X is one of the following: (i) $CP_{r-1} \times Q_r$; (ii) $P(T^*CP_r)$; or (iii) CP_{2r-1} blown up along CP_{r-2}.*

Here $Q_r \subset CP_{r+1}$ denotes the r-quadric— that is, r-dimensional compactified complexified Minkowski space— while the projectivization of a bundle $E \to Y$ is defined by $P(E) := (E - 0_Y)/(C - 0)$.

On the other hand, spaces (i) and (iii) aren't complex contact manifolds, as follows most easily seen from the fact that that $\Gamma(CP_{r-1}, \Omega^1(1)) = 0$, so that the obvious foliations by CP_{r-1}'s would necessarily have Legendrian leaves, so that these spaces would then have to be of the form $P(T^*Y)$, where Y is the leaf space Q_r or CP_r; contradiction. So the only candidate for a twistor space is (ii), which is indeed the twistor space of $G_2(C^{r+1})$. Theorem 1 now follows from the Leray-Hirsch theorem on sphere bundles: $b_2(Z) = b_2(M) + 1$.

In dimension 4, Theorem 1 contains Hitchin's result [2], since a compact self-dual manifold with $b_2 = 0$ must be conformally flat. On the other hand, there are already *two* symmetric examples with $b_2 = 0$ dimension 8, so the Poon-Salamon result [4] really contains further information about this case.

Is there a more elementary proof of Theorem 1? Haiwen Chen *(private communication)* has given a Weitzenböck argument for a weaker version of this theorem, but his proof unfortunately requires the hypothesis of positive *sectional* curvature.

Acknowledgements. The author would like to thank Shigeru Mukai and Janoš Kollár for their helpful explanations of the theory of Fano manifolds.

References

[1] T.N. Baily and M.G. Eastwood, *Complex Paraconformal Manifolds— Their Differential Geometry and Twistor Theory,* Forum Math., *to appear.*

[2] N.J. Hitchin, *Kählerian Twistor Spaces,* Proc. Lond. Math. Soc. 43 (1981) 133-150.

[3] S. Mori, *Cones of Curves and Fano Manifolds,* Proc. Int'l. Congr. Math. 1983.

[4] Y.-S. Poon and S.M. Salamon, *Eight-Dimensional Quaternionic-Kähler Manifolds with Positive Scalar Curvature,* J. Diff. Geometry , *to appear.*

[5] S.M. Salamon, *Quaternionic-Kähler Manifolds,* Inv. Math. 67 (1982) 143-171.

[6] J.A. Wiśniewski, *On Fano Manifolds of Large Index,* preprint, Warsaw University, 1990.

Homogeneity of Twistor Spaces

Andrew F. Swann

Let Z be a twistor space of a quaternionic Kähler manifold M of positive scalar curvature. If Z is compact and homogeneous, then it is known that M is symmetric [1]. In this note we will be concerned with the case when Z is not necessarily compact but is still homogeneous.

Such a twistor space is a complex contact manifold of dimension $2n+1$, so possesses a line bundle $\mathcal{L}$ and a holomorphic 1-form $\theta \in \Omega^1(\mathcal{L})$ such that $\theta \wedge (d\theta)^n$ vanishes nowhere. A holomorphic vector field on Z is an infinitesimal contact transformation if it preserves θ up to scale. We say Z is a homogeneous twistor space if the connected component G of the identity of the group of holomorphic contact transformations of Z acts transitively.

Lichnerowicz [2] shows that homogenous complex contact manifolds are related to coadjoint orbits. The construction is just derived from that for homogeneous symplectic spaces. Let H be the stabiliser of a point and let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. We have an exact sequence

$$0 \longrightarrow \ker\theta \longrightarrow T'Z \xrightarrow{\theta} \mathcal{L} \longrightarrow 0,$$

so θ^* gives an inclusion of $\mathcal{L}^*$ into $T'^*Z \cong \operatorname{Ann}\mathfrak{h} \subset \mathfrak{g}^*$. If π is the projection $\mathcal{L}^* \to Z$, then we may define a holomorphic 1-form α on $\mathcal{L}^*$ by

$$\alpha_b(v) = b(\theta(\pi_* v)) = (\theta^* b)(\pi_* v),$$

for $v \in T_b\mathcal{L}^*$ and $\omega = d\alpha$ is then a complex symplectic form on $\mathcal{L}^*$. Now any holomorphic contact transformation of Z lifts to a transformation preserving α and hence ω, so if $\tilde{X}$ is the lift of a vector field X we have

$$\tilde{X} \lrcorner \omega = \tilde{X} \lrcorner d\alpha = L_{\tilde{X}}\alpha - d(\tilde{X} \lrcorner \alpha) = -d(\tilde{X} \lrcorner \alpha),$$

showing that the immersion $\mathcal{L}^* \to \mathfrak{g}^*$ is just a moment map, up to sign. Note that the moment map commutes with action of the scalars on the fibres of $\mathcal{L}^*$. Now, in general, if μ^X is a moment map for a vector field X, then the moment map $\mu^{[X,Y]}$

of the bracket of two vector fields differs from the Poisson bracket $\{\mu^X, \mu^Y\}$ by a constant. However, in our case, both these expressions are linear functions fibrewise, so the constant must be zero (cf. [3]). Thus the image of the moment map is a union of orbits for coadjoint action of G on $\mathfrak{g}^*$.

In the case of homogeneous complex contact manifolds there are now two cases: either G acts transitively on $\mathcal{L}^* \setminus 0$ or it does not. However, if Z is a twistor space then $\mathcal{L}^* \setminus 0$ is a hyperKähler manifold whose tangent space contains the lift of $\ker\theta$ as quaternionic subspace [4]. For a given point of $\mathcal{L}^* \setminus 0$ we may now use the action of the complex structures J and K on the tangent space of $\mathcal{L}^* \setminus 0$ to obtain a holomorphic vector field generating the action of the scalars through that point, so the image of the moment map consists of just one orbit (which contains zero in its closure).

If G is reductive, then we may choose an invariant inner product and identify $\mathfrak{g}^*$ with $\mathfrak{g}$. The image of $\mathcal{L}^* \setminus 0$ is now a nilpotent orbit in the semi-simple part of $\mathfrak{g}$. However, it was shown in [4], that the projectivisation of a nilpotent orbit is the twistor space of a quaternionic manifold of positive scalar curvature. To summarise:

If Z is a twistor space of a quaternionic Kähler manifold of positive scalar curvature such that Z is homogeneous as a complex contact manifold and the symmetry group G is reductive, then Z is the projectivisation of a nilpotent orbit of the semi-simple part of $\mathfrak{g}$, up to finite covers.

Acknowledgements. I would like to thank C.R. Lebrun for useful discussions.

References

[1] Boothby, W.M., 1961, *Homogeneous Complex Contact Manifolds,* Proc. Symp. Pure Math., **3**, 144–154; 1962, *A Note on Homogeneous Compex Contact Manifolds,* Proc. AMS, **13**, 276–280; Wolf, J.A., 1965, *Complex Homogeneous Contact Manifolds and Quaternionic Symmetric Spaces,* J. Math. Mech., **14**, 1033–1047.

[2] Lichnerowicz, A., 1986, *Espaces homogènes de contact complexes et espaces homogènes localemant conformément symplectiques complexes hamiltoniens,* C.R. Acad. Sci. Paris, **302**, 395–399; 1988, *Variétés de Jacobi et espaces homogènes de contact complexes,* J. Math. Pures Appl., **67**, 131–173.

[3] Arnol'd, V.I., "Mathematical Methods of Classical Mechanics," GTM **60**, Springer.

[4] Swann, A.F., 1991, *HyperKähler and Quaternionic Kähler Geometry,* to appear in Math. Annalen.

$SL(2)_q$ – Spin Networks

by Louis H. Kauffman

The original spin networks of Roger Penrose [3] are based on $SL(2)$ - invariant tensors. As explained in [1], the binor calculus can be seen as a special case of the bracket model of the Jones polynomial. In fact, this viewpoint extends to a generalization of spin nets corresponding to the re-coupling theory for the quantum group $SL(2)_q$. (See [2].) It is the purpose of this note to indicate the basic ingredients in this generalization.

First, $q = \sqrt{A}$ as in [1]. We replace the binor identity with the bracket identity

$$\times = A \asymp + A^{-1})($$

and loop value $O = -A^2 - A^{-2} = \delta$. The result of expanding any link diagram is then its bracket polynomial. For example

$$\infty = A \, \infty + A^{-1} \, O \, O$$

$$= (A + A^{-1}(-A^2 - A^{-2}))\delta$$

$$\infty = -A^{-3}\delta \,,$$

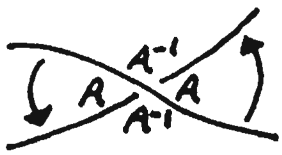
and

$$\text{OO} = A^{-1} \, \text{OO} + A^{+1} \, \text{OO}$$

$$= A^{-1}(-A^{-3})\delta + A^{+1}(-A^{3})\delta$$

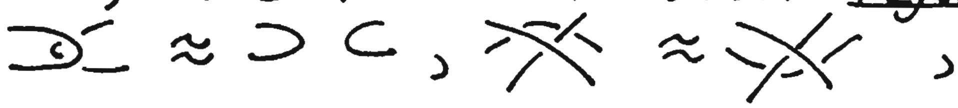
$$\text{OO} = (-A^{-4} - A^{4})\delta \,.$$

Note the convention for using this generalized binor identity:



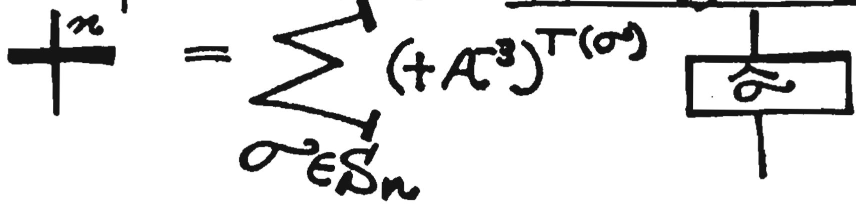
Turn the over-crossing line counterclockwise. Label the regions swept out as A. Label the remaining two regions as $B = A^{-1}$. In the expansion

Evaluations of link diagrams via the bracket identity are invariant under <u>regular isotopy</u>:



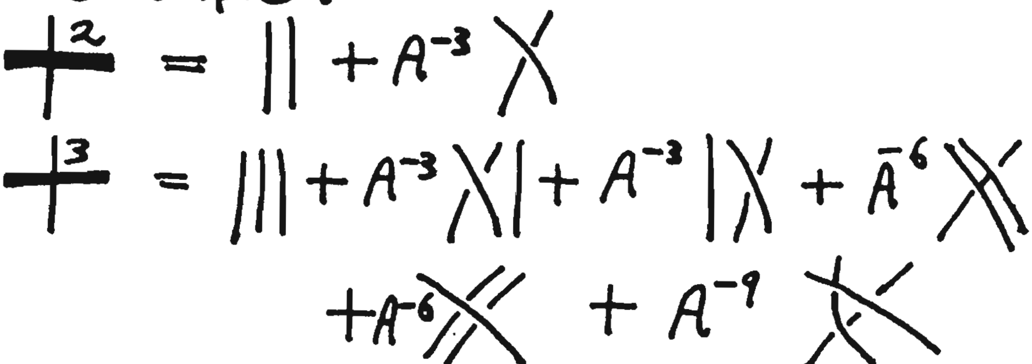
and satisfy the following rules for curls:

<u>Now</u>, replace the <u>anti-symmetrizer</u> by



where $\hat{\sigma}$ is a minimal braid projecting to the permutation σ (all crossings of type A: ╳), and $T(\sigma)$ denotes the minimal number of transpositions required to write σ.

For example:



Note that at $A=-1$ we recover the binor identity, and the usual anti-symmetrizers. The basic property of an anti-symmetrizer is that 1)

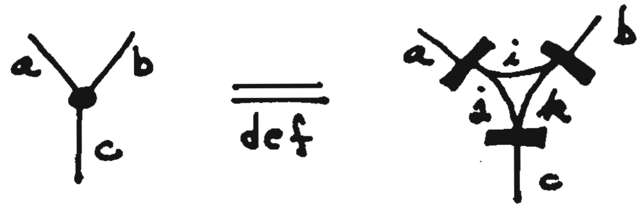
2)

Here both properties are true, with an appropriate generalization of the factorial:

$$n! = \sum_{\sigma \in S_n} (A^{-4})^{T(\sigma)} = \prod_{k=1}^{n} \left(\frac{1-A^{-4k}}{1-A^{-4}} \right).$$

Note how this works:

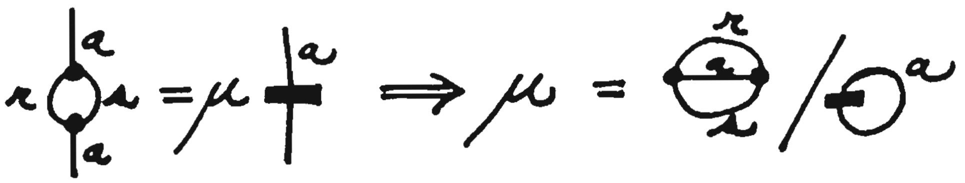
Having defined anti-symmetrizers, we now can define 3-vertices:

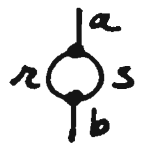


$i+j=a$
$j+k=c$
$i+k=b$,

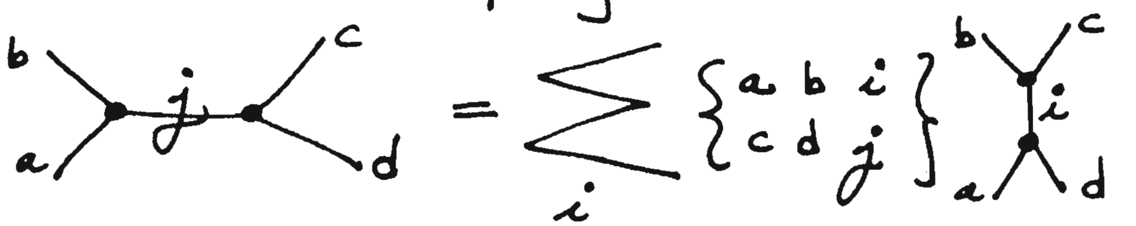
admissible when a, b, c satisfy the triangle inequality, and $a+b+c$ is even.

The usual apparatus of recoupling theory now generalizes and each quantity can be expressed as a q-spin network evaluation. For example,



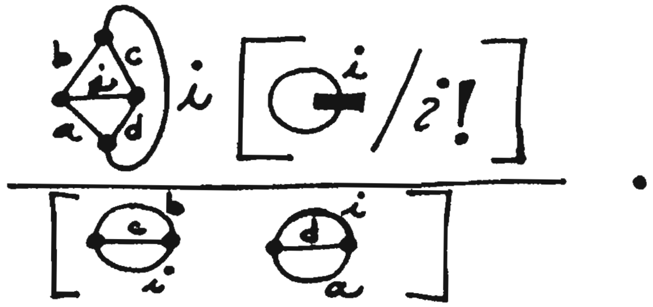
and  $= \phi$ if $a \neq b$ (ϕ denotes zero.).

Thus we can define q-6j symbols via the re-coupling



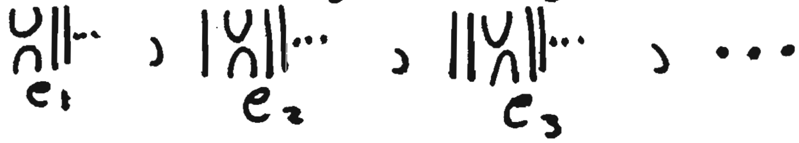
Then, taking traces, we find

$$\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix} =$$

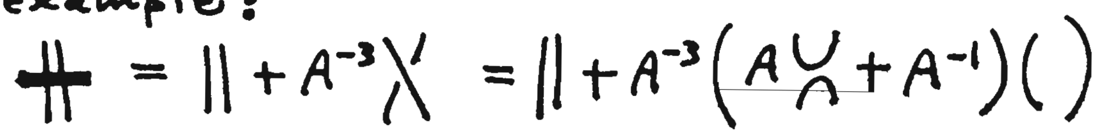


Specific formulas for these $q-6j$ coefficients can then be obtained just as in the chromatic method for classical spin networks.

Finally, it is worth noting that each anti-symmetrizer can be written in expanded form as an element in the Temperley-Lieb algebra generated by



For example:



$$= 1 + A^{-3}(Ae_1 + A^{-1})$$
$$= (1 + A^{-4}) + A^{-2}e_1$$
$$= (1 + A^{-4})[1 + \delta^{-1}e_1]$$
$$= (2!)[1 + \delta^{-1}e_1].$$

In general,  $= (n!) f_n$ where

$e_i f_n = 0$ for $i \leq n$ and $f_n^2 = f_n$.

In the next installment, we shall discuss the fate of the Spin Geometry Theorem in this context.

References

1. L. Kauffman, Spin networks and the Jones polynomial. Twistor Newsletter No. 29, November 1989.
2. L. Kauffman, Knots, spin networks and 3-manifold invariants. (to appear).
3. R. Penrose. Angular momentum – an approach to combinatorial space-time. from Quantum Theory and Beyond ed. by T. Bastin. Camb. Univ. Press (1979).

Dept. of Math. Stat. Comp. Sci.
Univ. of Ill. at Chicago
Chicago, Ill. 60680

L. Mason: A new programme for light cone cuts and Yang-Mills holonomies

Appendix: explicit coordinate expressions

One of the main features of the Kozameh-Newman formalism is the use of the 4 functions $Z^{AA'} = \eth^A \eth^{A'} Z$ at some fixed $\pi_{A'}, \bar\pi_A$ as coordinates on $\mathcal{M}$. The 0^{th} and 1^{st} derivatives of Z are determined from $Z^{AA'}$ by $Z = \pi_{A'}\bar\pi_A Z^{AA'}$ and $Z^{A'} = \bar\pi_A Z^{AA'}$ (these follow from the homogeneity of Z) so $Z^{AA'}$ is the part of the second jet of Z as a function of $\pi_{A'}, \bar\pi_A$ containing only the mixed second derivatives. In flat space Z can be taken to be $Z = x^{AA'}\pi_{A'}\bar\pi_A$ where $x^{AA'}$ are affine coordinates on Minkowski space, so $Z^{AA'} = x^{AA'}$.

Note that if a quantity f has homogeneity n in $\pi_{A'}$, then $\pi_{A'}\eth^{A'}\eth^{A'_1}\ldots\eth^{A'_n} f = 0$ by homogeneity so that $\eth^{A'}\eth^{A'_1}\ldots\eth^{A'_n} f = \pi^{A'}\pi^{A'_1}\ldots\pi^{A'_n}\eth^{n+1} f$ for some quantity $\eth^{n+1} f$ of weight $-n-2$. Transfering $\Lambda = \eth^2 Z$, and $\Lambda^A = \eth^A\eth^2 Z$ to the $Z^{AA'}$ coordinate system, we find that $(*)$ becomes:

$$0 = g^{A(A'B')B} + \bar\pi^A\bar\pi^B\bar\pi_D\partial_{\underline{c}}\bar\Lambda^{(A'}g^{B')D\underline{c}} + \pi^{A'}\pi^{B'}\pi_{D'}\partial_{\underline{c}}\Lambda^{(A}g^{B)D'\underline{c}}$$

$$+\tfrac{1}{2}\pi^{A'}\pi^{B'}\bar\pi^A\bar\pi^B(\pi_{C'}\bar\pi_C\partial_{\underline{d}}\eth^2\bar\eth^2 Z + \partial_{\underline{c}}\Lambda\partial_{\underline{d}}\bar\Lambda)g^{\underline{cd}}$$

where $\underline{c}$, $\underline{d}$ are the concrete indices associated to the $Z^{AA'}$ coordinate system; $\underline{c} = CC'$ etc.. If we adjoin to this the equation $g^{A[A'B']B} = \Omega^2\varepsilon^{AB}\varepsilon^{A'B'}$ where Ω is the undetermined conformal factor, one can solve for g^{ab} provided $(\pi^{A'}\bar\pi^A\partial_{AA'}\Lambda\ \pi^{A'}\bar\pi^A\partial_{AA'}\bar\Lambda) \neq 1$.

A new programme for light cone cuts and Yang-Mills holonomies

In a series of papers Kozameh & Newman have developed formalisms for the study of general space-times and Yang-Mills fields that generalize many of the ingredients of the nonlinear graviton construction and Ward construction to the non self-dual equations. An important motivation for the study of these formalisms is that it seems likely that these structures will have to play a role in any twistorial understanding of the full vacuum and Yang-Mills equations even if only as an intermediate stage in some more elaborate framework.

To date, however, the incorporation of the full vacuum equations into the light cone cut formalism has been problematic even in principle and it has only been possible to do so by introducing substantial additional structure (the holonomy of one of the spin connections as an additional separate connection satisfying its own equations) Kozameh, Lamberti & Newman (1990). The purpose of this note is to present a new strategy that gives a clean articulation of the vacuum equations and more generally the Bach equations within the framework. The articulation of the field equations reduces to one scalar differential equation (and some boundary conditions). Unfortunately the calculations required to obtain an explicit form for the main equation have so far proved intractable except in the linearized and self-dual case. Nevertheless certain basic features are clear from its derivation and linearized form.

The basic field equations for the Yang-Mills formalism have already been obtained in two forms in Kozameh & Newman (1985). However, the new strategy yields a new point of view on these equations that eliminates various steps in Kozameh & Newman's approach and shows that only one scalar equation needs to be considered. The Bach equations are in fact the Yang-Mills equations for the local twistor connection so there is more contact between the two formalisms than one might have originally supposed.

The proposed strategies are not yet fully worked out and in the fully nonlinear cases various difficulties remain, so the basic idea will be illustrated by the linearized version of the theory where the ideas all work. In this note I shall use homogeneous coordinates that lead to simplifications in the derivation of many of the formulae. In the case of gravitation I also derive the formalism relative to a space-like hypersurface—asymptotic simplicity is inessential.

§1 The basic formalism. The formalism for curved space-times is based on an identification of the space of scaled null geodesics $\mathcal{N}$ with $T^*\mathcal{O}(1,1)$, where $\mathcal{O}(1,1)$ is the line bundle of homogeneity degree $(1,1)$ functions on the sphere. In the usual presentation of the formalism, $\mathcal{O}(1,1)$ is past null infinity and a scaled null geodesic is represented by a 1-form orthogonal to it at its intersection point with null infinity. The identification can also be made by identifying the light cone of a finite point of $\mathcal{M}$ with $\mathcal{O}(1,1)$ and $\mathcal{N}$ with its cotangent bundle.

The bundle $\mathcal{O}(1,1)$ is coordinatized by $(u,\pi_{A'},\bar{\pi}_A)$ which are taken to be homogeneous coordinates, $(u,\pi_{A'},\bar{\pi}_A)\sim(\lambda\bar{\lambda}u,\lambda\pi_{A'},\bar{\lambda}\bar{\pi}_A)$, $\pi_{A'}\neq 0$. If $u=Z(\pi_{A'},\bar{\pi}_A)$ is a section of $\mathcal{O}(1,1)$, then it determines the cotangent vector $(\partial^{A'}Z)d\pi_{A'}+(\partial^{A}Z)d\bar{\pi}_A$ where $\partial^{A'}=\partial/\partial\pi_{A'}$, $\partial^{A}=\partial/\partial\bar{\pi}_A$ and homogeneity implies that $Z=\pi_{A'}\partial^{A'}Z=\bar{\pi}_A\partial^{A}Z$. The cotangent bundle of $\mathcal{O}(1,1)$ can therefore be coordinatized by $(Z^A,\pi_{A'},Z^{A'},\bar{\pi}_A)$ subject to the relation $\pi_{A'}Z^{A'}=\bar{\pi}_AZ^A$. (These coordinates, strictly speaking, involve an additional phase—we are really dealing with the space of null geodesics with a complex scale, that is a parallel propagated spinor aligned along the null geodesic rather than just a covector.)

Remark: One can also produce such an identification relative to a spacelike hypersurface $\mathcal{H}$. All the spinor indices in the following should be taken to be concrete. Choose coordinates $x^{AA'}$ on a space-time $\mathcal{M}$ such that for some fixed $T_{AA'}$, $\mathcal{H}$ is given by $T_{AA'}x^{AA'}=0$. Let $\pi_{A'}$ be homogeneous coordinates on $\mathbb{CP}^1$. Define $Z(x,\pi_{A'},\pi_A)=x^{AA'}\pi_{A'}\bar{\pi}_A$. The space of scaled null geodesics is canonically identified with $T^*\mathcal{H}$. We define a map from $T^*(\mathcal{O}(1,1))$ to $T^*\mathcal{H}$ by identifying the point $(Z^A,\pi_{A'},Z^{A'},\pi_A)$ in $T^*(\mathcal{O}(1,1))$ with the cotangent vector $\pi_{A'}\bar{\pi}_A dx^{AA'}$ at the point $x^{AA'}$ whose coordinates are determined by the relations $T_{AA'}x^{AA'}=0$, $x^{AA'}\pi_{A'}=Z^A$ and $x^{AA'}\bar{\pi}_A=Z^{A'}$ (these are only 4 relations as $\pi_{A'}Z^{A'}=\bar{\pi}_AZ^A$). The coordinates $(Z^A,\pi_{A'},Z^{A'},\pi_A)$ can be thought of as being coordinates on the spin bundle of $\mathcal{M}$ restricted to $\mathcal{H}$ (although for this one must choose an arbitrary identification of phase of $\pi_{A'}$ with that of the spinors) and can be continued to be functions on the spin bundle of $\mathcal{M}$ by requiring that they be constant along the null geodesic spray. Note that the identification of the $\pi_{A'}$ with the spinors at each point cannot be made to be holomorphic without substantial restriction on the curvature.

Clearly, such an identification encodes little of the space-time geometry. The conformal structure is encoded by the knowledge, for each x in $\mathcal{M}$, of the S^2 in $P\mathcal{N}$ of light rays incident with x. This can be represented by a 'cut function' $Z(x,\pi,\bar{\pi})$ which yields the S^2 of light rays incident with x as $(Z^A,\pi_{A'},Z^{A'},\bar{\pi}_A)=(\partial^AZ,\pi_{A'},\partial^{A'}Z,\bar{\pi}_A)$ in $T^*(\mathcal{O}(1,1))$. The cut function should be thought of as a generating function for these S^2's—the S^2's are always regular, whereas Z may well be singular. The cut function is the basic variable in the formalism.

§2 The reconstruction of the conformal structure from the cut function. This relies only on the property that, for each $\pi_{A'}$, Z is constant along a foliation by null hypersurfaces and that, for fixed x, as $\pi_{A'}$ varies, these surfaces vary through all null hypersurface elements at x (this follows from the fact that holding Z and $\pi_{A'}$ and $\bar{\pi}_A$ constant gives a Legendrian submanifold in $P\mathcal{N}$ and hence a null hypersurface in $\mathcal{M}$). Let ∂_a denote the coordinate derivative on $\mathcal{M}$ holding $\pi_{A'}$ constant. Then the conformal metric on 1-forms is determined by the condition $g^{ab}\partial_a Z\,\partial_b Z = 0$ as $\pi_{A'}$ and $\bar{\pi}_A$ vary. (Here the indices $a,b\,\ldots$ are regular tangent space indices to $\mathcal{M}$.) If an arbitrary Z has been given (as we shall often suppose) as π and $\bar{\pi}$ vary, $\partial_a Z$ will determine a 'crinkly cone' in $T^*\mathcal{M}$ that will not be quadratic and will give rise to a conformal Lorentzian analogue of a Finslerian structure.

At a fixed value of π and $\bar{\pi}$ one can obtain an expression for the metric that best approximates the crinkly cone at that value of π and $\bar{\pi}$ in terms of derivatives of Z and an arbitrary conformal factor by observing that if g^{ab} were independent of $\pi_{A'}$ and $\bar{\pi}_A$, then $g^{ab}\partial_a Z\,\partial_b Z = 0$ would imply

$$g^{cd}\partial^A\partial^B\partial^{A'}\partial^{B'}(\partial_c Z\,\partial_d Z) = 0 \tag{1}$$

This is 9 equations on 10 unknowns and therefore determines a 1-dimensional ray in the space of metrics, that is to say a conformal structure. This ray will in general vary with π and $\bar{\pi}$. The more explicit formulae obtained by Kozameh & Newman can usefully be written in this notation as well. This is a convenient framework for explicit calculations. However, it is complicated, and is not essential for an understanding of the programme.

We will also be interested in the 'non-Finslerian' condition—if a general choice of Z is made, the conformal structure determined by equation (1) will depend on $\pi_{A'}$ and $\bar{\pi}_A$. By taking an additional ∂^A derivative of (1) we see that the conformal structure will be independent of $\bar{\pi}_A$ iff $g^{ab}(\partial^{A'}\partial^{B'}\bar{\eth}^3(\partial_a Z\,\partial_b Z)) = 0$. The complex conjugate equation can be imposed to ensure independence from $\pi_{A'}$ also, although if everything is global in $\pi_{A'}$, g^{ij} will be global and holomorphic in $\pi_{A'}$ and therefore constant by Liouville's theorem. It will be shown in the next section that the weaker necessary scalar condition

$$g^{ab}\eth^3\bar{\eth}^3(\partial_a Z\,\partial_b Z) = 0 \tag{2}$$

is sufficient in linearized theory as this condition implies that g^{ab} is harmonic on the sphere so that the maximum principle holds—the only solutions should be conformal structures that are constant on the $(\pi_{A'}, \bar{\pi}_A)$ sphere. This is probably also sufficient in the full nonlinear regime also. The condition that actually arises from the main field equation is different so this condition will not be studied any further.

§3 The main field equation. It is difficult to impose the vacuum field equations directly as the Ricci tensor depends on Ω as well as Z. The strategy I shall follow is to impose the Bach equations, $B_{ab}=0$ instead. These equations are conformally invariant and so constrain only Z. They are obtained from the conformally invariant Lagrangian $\int C^a{}_b \wedge {}^*C^b{}_a$ where $C^a{}_b = C^a{}_{bcd}dx^c \wedge dx^d$ is the Weyl tensor. The Bach tensor $B_{ab} = \nabla^c\nabla^d C_{acbd} - \frac{1}{2}R^{cd}C_{acbd}$ is symmetric, trace free, divergence free and vanishes when the Ricci tensor does. The Bach equations are a fourth order set of hyperbolic equations that lead to a unique evolution of initial data. If the initial data is constructed from a vacuum evolution of vacuum initial data, then the uniqueness implies that the vacuum evolution must agree with the Bach tensor evolution. Thus, if the Bach equations are imposed, the restriction to the vacuum equations can be implemented by choice of boundary conditions or initial data. In particular, if the space-time is obtained from Bach tensor evolution of the standard data at null infinity it must necessarily be conformal to vacuum. The main field equation will be the scalar equation

$$l^a l^b B_{ab}(x, \pi_{A'}, \bar{\pi}_A) = 0 \tag{3}$$

where $B_{ab}(x,\pi,\bar{\pi})$ is the Bach tensor as computed from the conformal structure determined by equation (1) at a fixed value of $(\pi,\bar{\pi})$, and $l_a = \partial_a Z(x,\pi,\bar{\pi})$. Clearly, as $\pi_{A'}$ and $\bar{\pi}_A$ vary we will obtain all the Bach equations when the conformal structure is independent of π and $\bar{\pi}$.

In linearized and half conformally flat theory this equation has the following remarkable features. The $l^a\partial_a$ derivatives can be integrated directly to yield an equation for Z as a function of π and $\bar{\pi}$ with the x coordinates merely parametrizing the solution space. The equation descends to an equation in $\mathcal{N}$ that, together with global considerations, determines which two spheres correspond to light cones of points of $\mathcal{M}$. Secondly, the solutions for Z, if global with the correct boundary conditions, lead to a conformal structure that is independent of π and $\bar{\pi}$ and hence satisfying the Bach equations.

This can be demonstrated in linearized theory as follows. In linearized theory $Z = x^{AA'}\pi_{A'}\bar{\pi}_A + z(x,\pi,\bar{\pi})$, $g^{ab} = \eta^{ab} + h^{ab}$ with z and h small and equation (1) reduces to:

$$h^{ab} = \partial^{A'}\partial^{B'}\partial^A\partial^B(Lz)$$

where $L = \pi^{C'}\bar{\pi}^{C}\partial_{CC'}$. Then a short computation shows that the main equation, (3), is just $L^5\eth^2\bar{\eth}^2 z = 0$. If the boundary conditions appropriate to the linearized vacuum equations are imposed, this can be integrated to give

$$\eth^2\bar{\eth}^2 z = \eth^2\bar{\sigma} + \bar{\eth}^2\sigma \tag{4}$$

where $\sigma = \sigma(x^{AA'}\bar{\pi}_A\pi_{A'}, \pi_{A'}, \bar{\pi}_A)$ etc. is the linearized asymtotic shear which is the free data for the field. This can be integrated directly by means of a Greens function to yield:

$$z(x,\pi,\bar{\pi}) = \oint |\pi_{A'}\pi'^{A'}|^2 \log|\pi_{A'}\pi'^{A'}|^2 \left(\eth^2\bar{\sigma} + \bar{\eth}^2\sigma\right)\pi'_{A'}d\pi'^{A'} \wedge \bar{\pi}'_A d\bar{\pi}'^A.$$

In order for this to yield an actual solution of the linearized vacuum equations we must have that h_{ab} as derived from the linearization of (1) is independent of π and $\bar{\pi}$. One can prove this in various ways, but the most suitable way for generalization is as follows.

Start with the main equation $L^5\eth^2\bar{\eth}^2 z = 0$. Act on this with the operator $N = \partial_{AA'}\partial^A\partial^{A'}$. It turns out that $[N,L^5] = \frac{5}{2}\Box L^4(\Upsilon + \bar{\Upsilon} + 6)$ where $\Upsilon = \pi_{A'}\partial^{A'}$ is the homogeneity operator and so the commutator vanishes on quantities of weight $(-3,-3)$ such as $\eth^2\bar{\eth}^2 z$. We also have the relation $N\eth^2\bar{\eth}^2 = \eth^3\bar{\eth}^3 L$ so that the main equation implies

$$\begin{aligned} 0 &= NL^5\eth^2\bar{\eth}^2 z = L^5 N\eth^2\bar{\eth}^2 z \\ &= L^5\eth^3\bar{\eth}^3(Lz). \end{aligned}$$

This integrates to yield $\eth^3\bar{\eth}^3(Lz) = 0$. However, $\eth\bar{\eth}h_{ab} = \pi_{A'}\pi_{B'}\bar{\pi}_A\bar{\pi}_B\eth^3\bar{\eth}^3(Lz)$ so this equation will imply that h_{ab} is harmonic on the $(\pi,\bar{\pi})$-sphere and so, by globality and the maximum principle it is independent of π and $\bar{\pi}$.

In the curved case, then, one expects that one can find a global solution $Z(x,\pi,\bar{\pi})$ to equation (3) given appropriate asymptotic data (to see existence, one needs only to use the asymptotic data to produce the space-time, and then use the space-time to produce Z). As a heuristic argument that this solution of equation (3) is unique with given boundary data, identify $(x,\pi,\bar{\pi})$ space with the total space of the spin bundle with coordinates $(x,\zeta_{\tilde{A}'},\bar{\zeta}_{\tilde{A}})$ in the natural way (the tilde over the spinor indices for the spin bundle is to emphasize that they are not the same as those introduced above and furthermore that they should be considered to be abstract). The operator N will be represented as $\nabla_{\tilde{A}\tilde{A}'}\partial^{\tilde{A}'}\partial^{\tilde{A}}$ where $\nabla_{\tilde{A}\tilde{A}'}$ is the horizontal lift of the space-time derivative, and $\partial^{\tilde{A}} = \partial/\partial\zeta^{\tilde{A}}$ etc. If we act on the main equation $l^a l^b B_{ab}(x,\pi_{A'},\bar{\pi}_A) = 0$ with N we obtain a scalar equation that is a necessary condition for the conformal structure to be non-Finslerian, since if the conformal structure is non-Finslerian, $Nl^a l^b B_{ab}(x,\pi_{A'},\bar\pi_A)=0$ as a consequence of $\nabla^a B_{ab}=0$. We have seen above in the linearized case, with the appropriate boundary conditions, that this condition is also sufficient. It remains to prove that this condition will be sufficient in the curved case also. If so, then the space-time must necessarily satisfy $B_{ab}=0$ and have the given asymptotic data and therefore by uniqueness of evolution of the Bach equations be unique. Note that the equation $Nl^a l^b B_{ab}(x,\pi_{A'},\bar\pi_A)=0$ on its own can be regarded as a kinematic equation that together with globality should lead to a non-Finslerian conformal structure—it is an identity, following from $\nabla^a B_{ab}=0$, for a genuine conformal structure.

§4 **The Yang-Mills case.** So far the explicit computation of equation (3) in terms of Z and its derivatives has proved intractable. However, the Yang-Mills analogue of the above ideas can be fully worked out in Minkowski space. Let $D_a=\partial_a-\gamma_a$ be a Yang-Mills connection on $\mathbf{M}$. The basic object of interest is the parallel propagator along light rays, a matrix valued funtion on the spin bundle of $\mathbf{M}$ $G(x,\pi,\bar\pi)$ satisfying $\pi^{A'}\bar\pi^A D_{AA'}G(x,\pi,\bar\pi)=0$. The connection is determined by the equation

$$\gamma^{AA'}=\partial^{A'}\partial^A(LG\circ G^{-1}) \tag{1'}$$

where $L=\pi^{A'}\bar\pi^A\partial_{AA'}$ as above. If G has been chosen arbitrarily, γ_a will depend on π and $\bar\pi$ as well as x. The main field equation is

$$\pi^{A'}\bar\pi^A D^b F_{ab}=0 \tag{2'}$$

in which F_{ab} is the curvature of the connection γ_a. This yields the following equation on G:

$$L^3\bar\eth J+[\bar J,L^3J]+3[L\bar J,L^2J]=0. \tag{3'}$$

where $J=G^{-1}\eth G$. In the Abelian case this can be integrated to give $\eth\bar\eth$ log $G=\bar\eth A+\eth\bar A$ where $A=A(x^{AA'}\pi_{A'}\bar\pi_A,\pi_{A'},\bar\pi_A)$ is part of the asymptotic connection.

The analogue of the non-Finslerian condition can be stated as $\eth\gamma_a=\bar\eth\gamma_a=0$. As before, if we have globality, this will be guaranteed by the weaker condition $\eth\bar\eth\gamma_{AA'}=\pi_{A'}\bar\pi_A\partial^2\bar\partial^2(LG\circ G^{-1})=0$ using the maximum principle for solutions of the Laplacian on S^2. The non-Finslerian condition that arises from equation (2') is obtained by acting on (2') with $D_{AA'}\partial^{A'}\partial^A$. Note that this vanishes automatically when γ_a is independent of π and $\bar\pi$ as a consequence of the Bianchi identity $D^aD^bF_{ab}=0$ so the vanishing of this quantity is a necessary condition if γ_a is to be non-Finslerian. In the linearized case it yields $L^3\eth^2\bar{\eth}^2(LG \circ G^{-1}) = 0$ which with suitable boundary conditions can be integrated to give the weaker version of the non-Finslerian condition. In the nonlinear case a more complicated equation is obtained, and its sufficiencey as a non-Finslerian condition has not yet been proved. Note that this equation if imposed on its own only constrains the γ_a dependence on π and $\bar{\pi}$, and there is no restriction on the x dependence.

§ 5 Conclusions and outlook: It seems likely that the main scalar equations, (3)and (3′) on Z and G are sufficient to encode the full Bach/Einstein equations and Yang-Mills equations. However, these equations are in effect articulated on the spin bundle of space-time, whereas the all important feature of the linearized and half flat cases is that the main field equation can be integrated directly to yield equations for G or Z on $\mathcal{N}$. Otherwise put, the equations of the linearized and half flat case yield precisely the condition that certain structures descend to $\mathcal{N}$ and that these structures can be used to write down the equations that determine Z and G. (In the half flat case the relevant structure is a complex structure.) It is this feature that leads to the twistor constructions. Unfortunately it is not even plausible that the main equations above will descend to $\mathcal{N}$ in this way, as it is easy to persuade oneself that if this were indeed possible, then the full Yang-Mills and Einstein equations would satisfy the Huygens property—that is, the solution at a point would depend only on the initial data at the intersection of the light cone at that point with the data surface. Instead, if the above ideas are to lead to a twistor construction for the full Yang-Mills and Einstein equation, there must be a further non-local transform to encode the structures on $\mathcal{N}$ or twistor space.

I would like to acknowledge many fruitful discussions with Kozameh & Newman and the hospitality of the physics department at the University of Pittsburgh where many of these ideas were first generated.

L.J. Mason

Kozameh & Newman (1983) Theory of light cone cuts of null infinity, J.Math. Phys., **24**, 2481.

Kozameh & Newman (1985) Yang-Mills equations and parallel propagation on closed paths, Phys. Rev.

D., **31**, #4, 801-809.

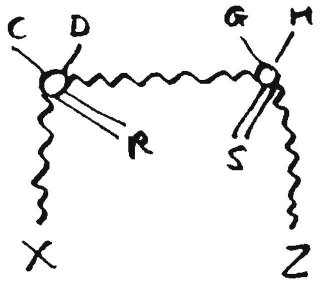
Kozameh, Lamberti & Newman (1990) Holonomy and the Einstein equations, Pittsburgh Preprint.

Appendix : see p. 14

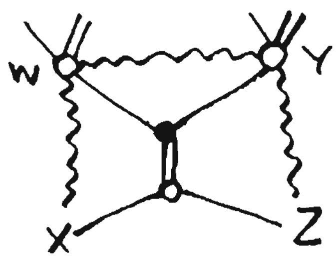
New contours with boundary, higher order diagrams, regularisation and massive propagators.

This note will further discuss and extend the "new" contours with boundary described in **TN30**. It turns out that these give the technical basis for a whole slew of advances in twistor diagram theory.

(1) An application of the spinor integral described in **TN30**. It follows from (2) on page 34 there, that there's a contour for the (projective) diagram

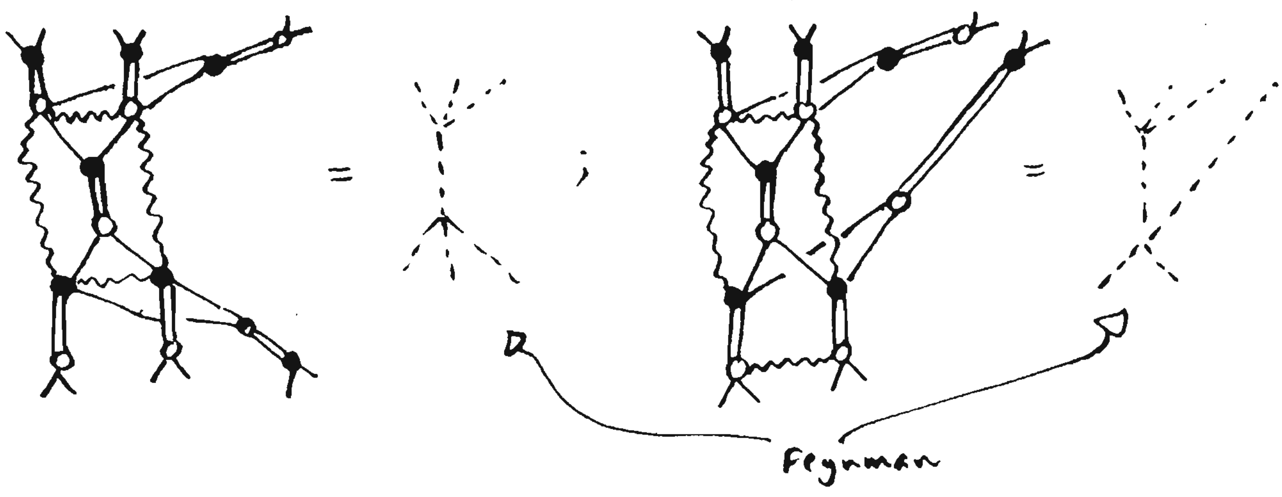


which allows CD = GH, R = S. This can be shown to induce a contour for



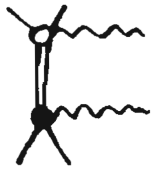
showing that the extra W. Z boundary line, as described in the article by L. J. O'D. in **TN 31**, is redundant.

(2) By an extension of this idea we can also produce contours for

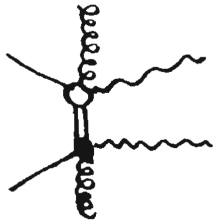


which makes some progress towards building higher order diagrams systematically.

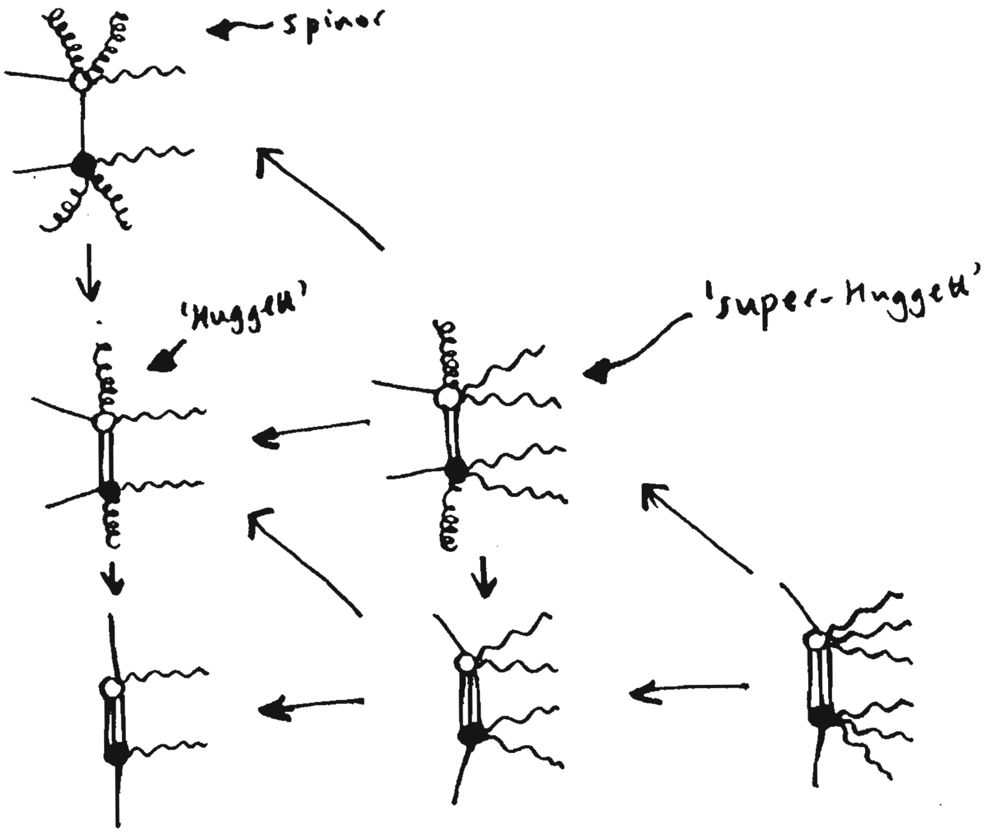
(3) In **TN30** I discussed explicit construction of the "Huggett" contour for the (projective) diagram



This can be thought of as



where the "springy" pole restricts from CP^3 to CP^2, so that the integral becomes a CP^2 analogy of the spinor integral just discussed. (The non-springy pole has the property of being inessential to the contour). Now this turns out to be just part of the following scheme of related integrals:

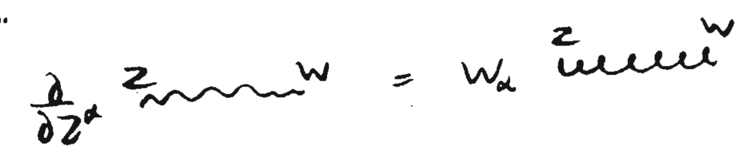


with relations:

↓ is just increase in dimension

← is "period"

↖ is through



Contours will also exist when the internal line is not a pole but a boundary.

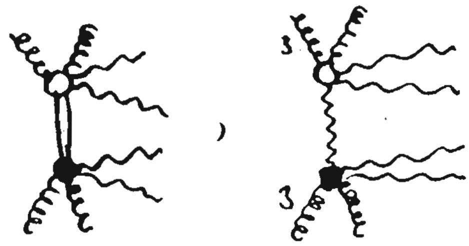
These cases are all of interest but the "super-Huggett" in CP^2 has a particular application. First I give an evaluation of it:

$$\oint_{\substack{\xi\cdot\alpha_1=0,\ \xi\cdot\alpha_2=0\\ \eta\cdot\gamma_1=0,\ \eta\cdot\gamma_2=0}} \frac{D^2\xi\wedge D^2\eta}{(\xi\cdot\beta)(\eta\cdot\delta)(\xi\cdot\eta)^2} = \left\{ \begin{array}{l} \displaystyle\int_0^1 du\int_0^1 dv \frac{\begin{smallmatrix}\alpha_1\alpha_2\\ \gamma_1\gamma_2\end{smallmatrix}\ \begin{smallmatrix}\alpha_1\\ \gamma_1\end{smallmatrix}}{\begin{smallmatrix}\alpha_1\alpha_2\\ \gamma_1\delta\end{smallmatrix}\ \begin{smallmatrix}\beta\alpha_1\\ \gamma_1\gamma_2\end{smallmatrix} - u\begin{smallmatrix}\alpha_1\alpha_2\\ \gamma_1\gamma_2\end{smallmatrix}\ \begin{smallmatrix}\alpha_1\beta\\ \delta\gamma_1\end{smallmatrix} - uv\begin{smallmatrix}\beta\\ \delta\end{smallmatrix}\ \begin{smallmatrix}\alpha_1\\ \gamma_1\end{smallmatrix}\ \begin{smallmatrix}\alpha_1\alpha_2\\ \gamma_1\gamma_2\end{smallmatrix}} \\ \\ \quad + 3 \text{ similar terms obtained by} \\ \qquad \alpha_1 \leftrightarrow \alpha_2,\quad \gamma_1 \leftrightarrow \gamma_2 \end{array}\right.$$

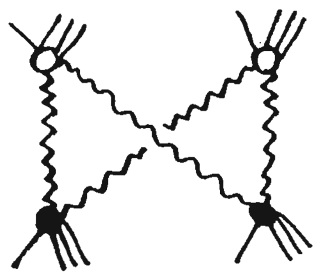
which can if desired be expressed as the sum of 8 dilogarithms. By thinking of C^2 as (CP^2 - line at infinity), we can apply this integral to

$$\oint_{\substack{x\cdot a=k_1,\ x\cdot b=k_2\\ y\cdot c=k_3,\ y\cdot d=k_4}} \frac{d^2x\wedge d^2y}{(x\cdot y-k)^2}$$

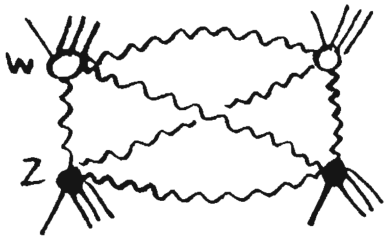
and so to the inhomogeneous twistor diagrams



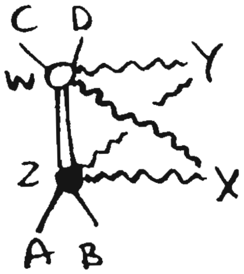
The point of this is that it allows a solution of the problem I described in **TN28**, that of finding a genuine contour with boundary for the inhomogeneous diagram



In that article I showed how the right answer for this diagram (which should be thought of as the *most fundamental* box diagram) could be obtained by a Pochhammer-contour construction involving the inhomogeneity in an essential way. This method *cheated,* however, in that I had to substitute logarithms for two of the boundary lines. I conjectured that this should not actually be necessary. The existence of the super-Huggett contour shows how we can avoid this cheat. Instead, what we need to do is to add in two extra inhomogeneous boundaries at infinity, i.e. consider



This diagram can be explicitly integrated to yield the right answer. The idea is to integrate out W_α and Z^α first. In fact I'll only give the result for the rather simpler case



i.e. $$\oint_{\substack{W.X=k\\ Y.Z=k\\ \underbrace{WY}=m,\ \overbrace{ZX}=m}} \frac{D^4W\wedge D^4Z}{\begin{smallmatrix}W\\|\\C\end{smallmatrix}\begin{smallmatrix}W\\|\\D\end{smallmatrix}\begin{smallmatrix}A\\|\\Z\end{smallmatrix}\begin{smallmatrix}B\\|\\Z\end{smallmatrix}\left(\begin{smallmatrix}Y\\|\\Z\end{smallmatrix}-k\right)^2}$$

because this has got just the same singularity structure in X^α and Y_α, while avoiding the complication due to the spin-1 fields. The result (obtained by applying the CP^2 integral formula given above) can be put in the form:

$$\frac{\pi^2}{6}-\mathrm{dilog}\left(\frac{-k\begin{smallmatrix}A\,B\\ \dagger\\ C\,D\end{smallmatrix}}{\begin{smallmatrix}Y\,A\,B\\ \dagger\dagger\dagger\\ X\,C\,D\end{smallmatrix}}\right)-\mathrm{dilog}\left(\frac{-m^2\begin{smallmatrix}A\,B\\ \dagger\\ C\,D\end{smallmatrix}}{k\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right.\begin{smallmatrix}A\,B\\ \dagger\\ C\,D\end{smallmatrix}}\right)$$

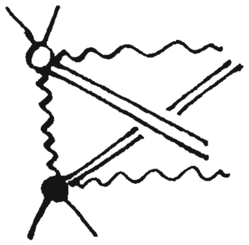
$$-\log\left(\frac{-k\begin{smallmatrix}A\,B\\ \dagger\\ C\,D\end{smallmatrix}}{\begin{smallmatrix}Y\,A\,B\\ \dagger\dagger\dagger\\ X\,C\,D\end{smallmatrix}}\right)\log\left(\frac{-\frac{m^2}{k}\begin{smallmatrix}A\,B\\ \dagger\\ C\,D\end{smallmatrix}}{\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right.\begin{smallmatrix}A\,B\\ \dagger\\ C\,D\end{smallmatrix}}\right)$$

To proceed with the integration, we take a loop from $Y.X = k$, round the branch point at

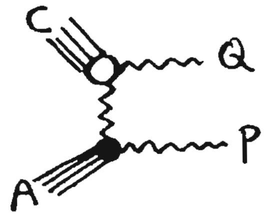
$$\begin{smallmatrix}Y\,A\,B\\ \dagger\dagger\dagger\\ X\,C\,D\end{smallmatrix}=-k\begin{smallmatrix}A\,B\\ \dagger\\ C\,D\end{smallmatrix}$$

and back again. All terms but the first dilogarithm contribute zero and the computation thereafter runs just as in the "cheating" method. Since the result is conformally invariant, it seems very likely that the added boundaries on $\underbrace{WY} = m$, $\overbrace{XZ} = m$ are in fact redundant and could be filled in by suitable "caps" to construct a contour with boundaries only as originally given. (However, the use of these extra boundaries is legitimate anyway according to current diagram philosophy, so this argument is not crucial).

Note: the article by L. J. O'D. in this **TN** describes the analogous construction in the rather simpler case of



(4) There is yet a further generalisation of these integrals. Note that the projective diagram



allows A. C = 0 (in fact the answer is then $\left(\dfrac{P.Q}{P.C\ A.Q}\right)^4$).

It follows that there is (in general position) a contour for the integral

$$\oint_{\substack{W.Z=0\\ W.Q=0,\ P.Z=0}} \frac{DWZ}{(W.C)^3 (A.Z)^3 \begin{matrix} W & A \\ Z & C \end{matrix}}$$

For when A. C = 0 this is just the same integral; now move the parameters and the contour must survive. Transferring this observation to the inhomogeneous context, this tells us that there are contours for integrals of form

$$\oint_{\substack{W.Z=k\\ \cdots}} \frac{D^4WZ}{(W.Z)} \cdots$$

which we have already noted are crucial for regularisation (see the end of my article in **TN 31**, and also L. J. O'D in this **TN**.) It *also* tells us there are contours for integrals of form

$$\oint_{\substack{\overset{\frown}{XZ}=m\\ \cdots}} \frac{D^4X\, D^4Z}{(\overset{\frown}{XZ})} \cdots$$

and this now turns out to yield a key new idea in the incorporation of *massive* fields into twistor diagram theory.

To see this, note that the heart of the problem is the representation of the Feynman propagator function, which (in the scalar case) satisfies

$$(\Box_x + m^2)\Delta_F(x-y;m) = \delta(x-y)$$

A formal solution of this equation is given by

$$\Delta_F(x-y;m) = \sum_1^{\infty} (m^2)^{n-1}\,\Box^{-n}\delta(x-y)$$

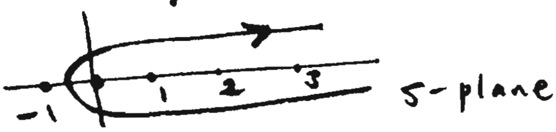
but of course the inverse derivatives are so far undefined.
To proceed I shall compose everything with test fields, i.e. perform

$$\iint d^4x\, d^4y \;((x-r)^2)^{-2}((y-s)^2)^{-2}\ldots$$

We have

$$\iint d^4x\, d^4y \;\frac{\delta(x-y)}{((x-r)^2)^2((y-s)^2)^2} = \frac{1}{((r-s)^2)^2}$$

and
$$\iint d^4x\, d^4y \;\frac{\Delta_F(x-y;m)}{((x-r)^2)^2((y-s)^2)^2} = \int \frac{(m^2(r-s)^2)^{s-1}\,\Gamma(-s)\Gamma(1-s)\,ds}{\sin\pi s}$$



This Barnes integral representation supplies a definitive specification of how the inverse derivatives are to be chosen: namely that the nth inverse derivative must correspond to the residue of this Barnes integral at $s = n-1$.

For instance it tells us that the $n = 1$ term is
$$\frac{\log(m^2(r-s)^2)+2\gamma}{(r-s)^2}$$

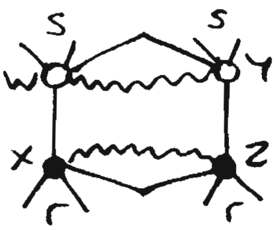
and the $n = 2$ term is
$$\frac{1}{2}\left[\log(m^2(r-s)^2)+2\gamma\right]^2 - \left[\log(m^2(r-s)^2+2\gamma-1\right] + \frac{\pi^2}{3}$$

These expressions can be thought of as regularisations of the divergent Fourier integrals

$$\int_{\substack{\text{timelike}\\ p^a}} d^4p \; \frac{e^{-ip.(r-s)}}{(p^2)^n}$$

We have earlier shown how infra-red and ultra-violet divergences can be regularised by inhomogeneous twistor diagrams. Can the same be done for these? Yes, it seems! The key idea is the combination of pole $(\overline{xz})^{-1}$ and boundary at $\{|\overline{xz}| = m\}$

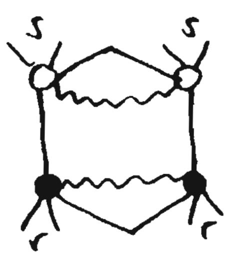
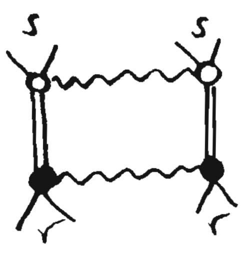
To see this explicitly, consider the diagram



where for temporary purposes I have indicated the factors $(\overline{xz})^{-1}$, $(\underline{wy})^{-1}$

by 

This formally satisfies

$\Box_r$  = 

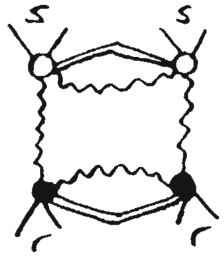
so if a contour exists such that the RHS integral yields $((r-s)^2)^{-2}$

then the LHS must correspond to a solution of $\Box_r^{-1}((r-s)^2)^{-2}$

In fact, such a contour does exist, and explicit evaluation gives

$$\frac{1}{(r-s)^2}\log\left(\frac{m^2}{k^2}(r-s)^2\right)$$

Similarly for n = 2, we can write down and then evaluate



$$= \frac{1}{2}\left[\log\left(\frac{m^2}{k^2}(r-s)^2\right)\right]^2 - \left[\log\left(\frac{m^2}{k^2}(r-s)^2\right) - 1\right] - \frac{\pi^2}{6}$$

(so that here the "new" contour with boundary plays an essential role.)

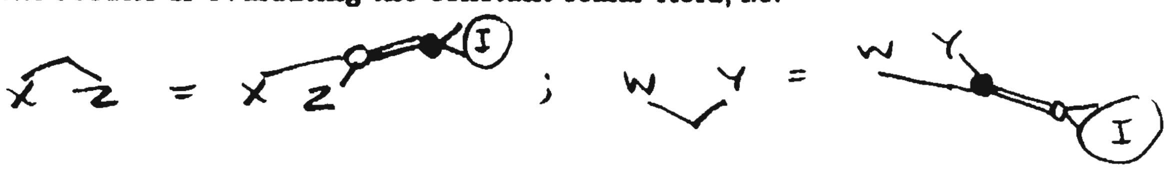
Obviously these results have general agreement in form with the expressions which sum to the Feynman propagator, but *exact* agreement implies something more. Firstly it suggests that we make a specific choice of the hitherto undetermined parameter k, namely

$$**** \qquad k = e^{-\gamma} \qquad ****$$

In fact this is essential if the m that enters into the specification of the boundary contour is to be equal to the m that appears in the coefficients of the power series.

Secondly, there remains the discrepancy of $\pi^2/2$. Examination shows that this can be removed provided we amend the prescriptions so that boundaries can lie on $\overset{\frown}{xz} = \pm m$, $\underset{\smile}{wy} = \pm m$. As yet this is only an *ad hoc* fix; underlying it is the fact that there are some questions of sign and time-direction which are closely involved in the definition of the Feynman propagator, and which have not yet been clarified. Furthermore it is not yet proved that that correct quantities can be generated for n = 3, 4, ... However, it seems very likely that there exists an integral scheme based on this contour structure which does allow the Feynman propagator (and *a fortiori*, the on-shell propagator) to emerge as a series of well-defined finite terms.

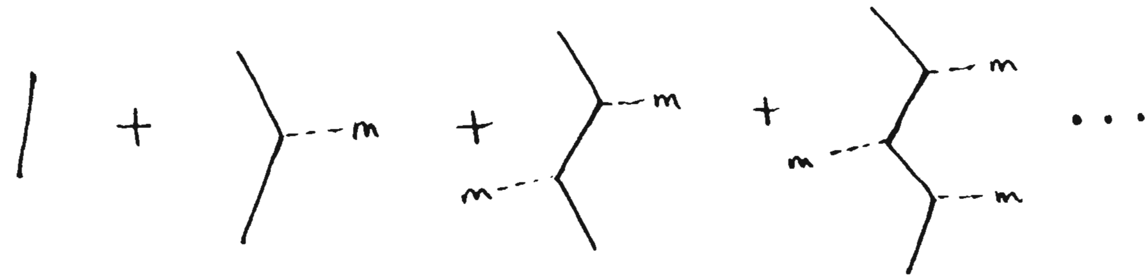
Note that the factors $(\overset{\frown}{xz})^{-1}$, $(\underset{\smile}{wy})^{-1}$, which play a key role here, are simply the results of evaluating the constant scalar field, i.e.



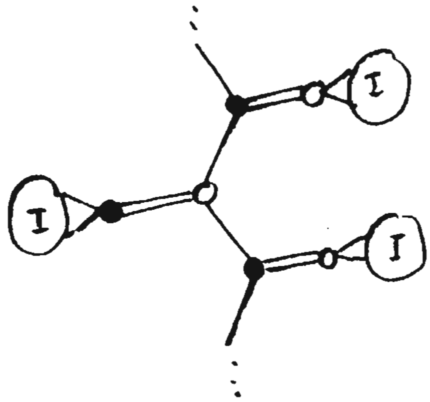
So the structure we are building here requires only the same ingredients as have been used in the description of massless field theory, with the sole addition of the constant field treated as an external field.

This suggests that the structure can be *directly* related to the "standard model" picture of massless fields acquiring mass through interaction with the constant Higgs field.

In the standard model the electron propagator is thought of as a sum over a series of (massless) Feynman diagrams of form



although this summation is purely formal, as none but the first of these has any finite meaning. But my prediction is that these divergent zigzag Feynman diagrams correspond systematically to perfectly finite twistor diagrams, each with a "skeleton" essentially of form

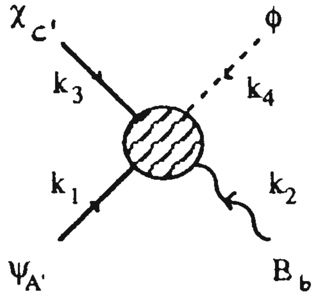


These should sum as an infinite series exactly to the correct massive propagator function. Much work still needed to show this though.

A.P.H. Andrew Hodges

A twistor construction for gauge potentials on Minkowski space

We construct a twistor expression (6) satisfying twistor analogues of (1), (2) in order to facilitate calculations of scattering amplitudes with exterior gauge potentials.
As an example let us consider the lowest-order contribution to the amplitude of the following process arising from the Standard Model:



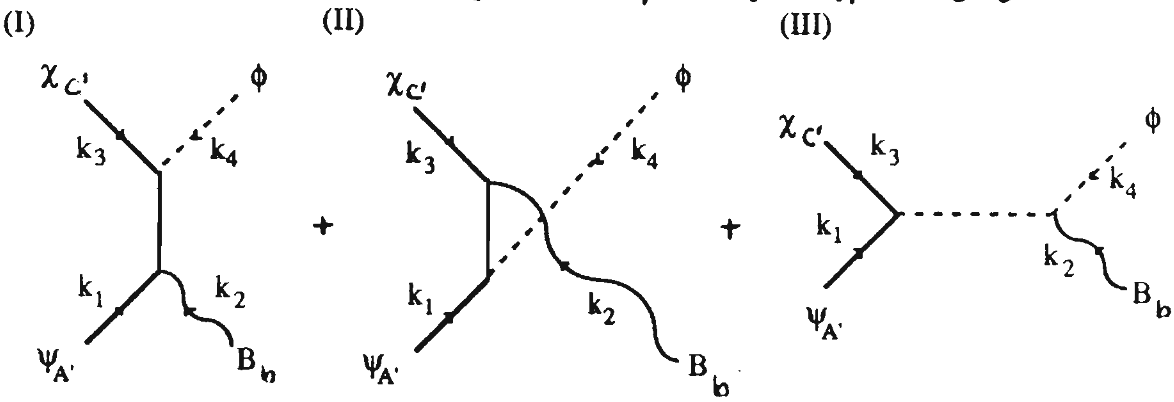
where $\psi_{A'}(k_1)$, $B_b(k_2)$ and $\bar{\chi}_C(-k_3)$, $\bar{\phi}(-k_4)$ are the (Fourier transforms of the) ingoing and outgoing (positive energy) fields, resp.. $B_b(k_2)$ is a U(1) gauge potential in Lorenz gauge

$$k_a B^a(k) = 0 \tag{1}$$

such that

$$F_{ab}(k) = \Theta_{(AB)}(k)\, \varepsilon_{A'B'} + \Psi_{(A'B')}(k)\, \varepsilon_{AB} = -ik_{AC'}\, B^{C'}{}_{B}(k)\, \varepsilon_{A'B'} - ik_{A'C}\, B^{C}{}_{B'}(k)\, \varepsilon_{AB} \tag{2}$$

(in momentum space), where $\Theta_{(AB)}$, $\Psi_{(A'B')}$ satisfy massless free-field equations.
Summing the contributing Feynman diagrams which, taken separately, are not gauge invariant



we get a momentum space integral (massless theory):

$$\int d^4k_1\, d^4k_2\, d^4k_3\, d^4k_4\, \delta^+(k_1)\, \delta^+(k_2)\, \delta^-(k_3)\, \delta^-(k_4)\, \delta(k_1+k_2+k_3+k_4)\ (\mathrm{I+II+III}) \quad \text{with}$$

$$\mathrm{I + II + III} = \phi(k_4)\, \chi_{C'}(k_3)\, B_{BB'}(k_2)\, \psi_{A'}(k_1)\, \Delta^{A'BB'C'} \tag{3}$$

up to some costant factors (coupling constants, i's, ...),

$$\text{where} \quad \Delta^{A'BB'C'} = \frac{(k_1+k_2)^{BC'}}{k_1 k_2}\, \varepsilon^{A'B'} + 2\,\frac{k_4^{BA'}}{k_1 k_4}\, \varepsilon^{B'C'} + \frac{k_4^{BB'}}{k_1 k_3}\, \varepsilon^{A'C'}$$

'=' after ∫-tion

$$\cong \frac{-k_4^{BC'}}{k_1 k_2}\, \varepsilon^{A'B'} + 2\,\frac{k_4^{BA'}}{k_1 k_4}\, \varepsilon^{B'C'} + \frac{k_4^{BB'}}{k_1 k_3}\, \varepsilon^{A'C'}$$

It then involves quite some spinor algebra, using

$$(k_1 + k_2 + k_3 + k_4)_a = 0\,,\quad k_{1A}{}^{A'}\psi_{A'}(k_1) = k_{3A}{}^{A'}\chi_{A'}(k_3) = 0 \qquad , \tag{4}$$

$$k_{2CB'}B^{B'}{}_B(k_2) = i\Theta_{(CB)}(k_2)\,,\; k_{2C'B}B^{B}{}_{B'}(k_2) = i\Psi_{(C'B')}(k_2) \quad , \tag{5}$$

to show how the above integral only depends on $\Theta_{(CB)}$, $\Psi_{(C'B')}$ and not on the gauge (cf. [1]). In fact, one finds that the term involving $\Psi_{(C'B')}$ vanishes (see (9)) and therefore one has helicity conservation (i.e. total incoming helicity = 0).

Since the relations (4) are already implicit in the algebra of a twistor diagram one is led to consider a (formal) twistor expression for $B_a(k_2)$ which also entails (1), (2) ($\Leftrightarrow$ (5)). As one has the correspondence of operators

$i\partial_{X AA'}$ (Minkowski space) $\leftrightarrow k_{AA'}$ (momentum space) $\leftrightarrow -X\partial_X$ (on generating functions on twistor space)

we represent

$$B_a(k) \leftrightarrow X\,V(\partial_X V)^{-1}\, f_{-4}(X^\alpha) + U\partial_X (XU)^{-1}\, f_0(X^\alpha) \tag{6}$$

where V_α, U^α are auxiliary twistors (corresponding to some gauge freedom) and $f_{-4}(X^\alpha)$, $f_0(X^\alpha)$ are twistor functions of homogeneity -4 , 0 resp. . Then

$$k_a B^a(k) \leftrightarrow XX\partial_X V(\partial_X V)^{-1} f_{-4}(X^\alpha) + \partial_X \partial_X XU(XU)^{-1} f_0(X^\alpha) = 0 \tag{7}$$

and

$$\Theta_{(AB)} = -ik_{AC'}B^{C'}{}_B \leftrightarrow iXX\,\partial_X V(\partial_X V)^{-1} f_{-4}(X^\alpha) + XU\,\partial_X\partial_X (XU)^{-1} f_0(X^\alpha) = \partial_X\partial_X\, i f_0(X^\alpha)$$

$$\Psi_{(A'B')} \qquad \leftrightarrow XX\, i f_{-4}(X^\alpha) \tag{8}$$

independent of V_α, U^α. This construction facilitates our calculations quite a bit. For example it is straightforward to verify that (3) = 0 for $F_{ab} = \Psi_{(A'B')}\varepsilon_{AB}$:

Let

$$\psi_{A'}(k_1) \leftrightarrow Z f_{-3}(Z^\alpha)\;,\quad \chi_{C'}(k_3) \leftrightarrow \partial_W\, g_{-1}(W_\alpha)\;,$$
$$B_{BB'}(k_2) \leftrightarrow XV(\partial_X V)^{-1} h_{-4}(X^\alpha)\;,\quad \phi(k_4) \leftrightarrow j_{-2}(Y_\alpha)$$

be representations of the exterior fields and potential by (dual) twistor functions. Then

$$\psi_A B_{BB'}\chi_{C'}\phi\Delta^{A'BB'C'} \leftrightarrow \Big\{ Z\,X\,V(\partial_X V)^{-1}\partial_W \Big[\sqcap\,\lfloor Y\,\rceil\partial_Y (Z\,X\,\partial_Z\partial_X)^{-1} + 2(-\rceil\partial_Y\lfloor Y)\sqcap(Z\,\partial_Y Y\partial_Z)^{-1} + \big[(-\lfloor Y\,\rceil\partial_Y\big](Z\,\partial_W W\partial_Z)^{-1} \Big]\Big\}_{contr} F$$

with F = fghj. The twistors outside are contracted into the [ ] - bracket from left to right in the obvious way.

$$= (\partial_X V)^{-1}\, VY\,[ZX\,\partial_W\partial_Y\,(ZX\,\partial_Z\partial_X)^{-1} - 2\,Z\partial_Y\,X\partial_W\,(Z\partial_Y\,Y\partial_Z)^{-1} - Z\partial_W\,X\partial_Y\,(Z\partial_W\,W\partial_Z)^{-1}]F$$

$=: DF$.

Operating on the single box this yields $D \underset{Z^\alpha \; X^\alpha}{\overset{W_\alpha \; Y_\alpha}{\bowtie}} =$

$$(\partial_X V)^{-1} V Y \left((\partial_Z \partial_X)^{-1} Z X \left(\bowtie - \bowtie \right) - 2(Y\partial_Z)^{-1} X Z \bowtie - (W\partial_Z)^{-1} X Z \bowtie \right)$$

$$(\partial_X V)^{-1} V Y Z X (W Y)^{-1} \left(\bowtie - 2\bowtie + \bowtie \right) = 0 \; . \qquad (9)$$

The amplitude corresponding to the a.s.d. part of F_{ab} leads to a twistor diagram independent of U^α corresponding to the spinor expression

$$\Delta^{A'BB'C'} \phi(k_4)\chi_{C'}(k_3) B_{BB'}(k_2)\psi_{A'}(k_1) = \frac{k_1k_2 - k_1k_3}{(k_1k_2)(k_2k_3)(k_3k_1)} k_{1E'}{}^{B} k_3{}^{DE'} \chi^{A'}\psi_{A'}\Theta_{DB}\phi \; .$$

Similarly in the non-abelian case a perturbative expansion in powers of coupling constants (g^n) relates the exterior free fields $\overset{o}{\Psi}_{(A'B')\Theta}{}^{\Phi}$, $\overset{o}{\Theta}_{(AB)\Theta}{}^{\Phi}$ of order g^0 in

$$\overset{o}{F}_{ab\Theta}{}^{\Phi} + g\overset{1}{F}\ldots = \varepsilon_{AB}\overset{o}{\Psi}_{A'B'\Theta}{}^{\Phi} + \varepsilon_{A'B'}\overset{o}{\Theta}_{AB\Theta}{}^{\Phi} + g\varepsilon_{AB}\overset{1}{\Psi}\ldots$$

$$= 2\partial_{[a}A_{b]\Theta}{}^{\Phi} - 2igA_{X[a}{}^{\Phi}A_{b]\Theta}{}^{X} \qquad (10)$$

linearly to the gauge potentials (again taken to satisfy the Lorenz gauge condition $\nabla_a A^a{}_\Theta{}^\Phi = \partial_a A^a{}_\Theta{}^\Phi = 0$). An analogous construction as in (6) can therefore be used.

One can apply this also to cases of higher helicities, such as for example in [2], as long as one is over flat space. If one looks at these potentials classically one has to ask, however, what their space-time version looks like (i.e. how they are to be 'contour-integrated') and they might turn out not to be very general.

[1] F. Müller (1991). Twistor Diagrams for some scattering amplitudes arising from the Standard Model. Qualifying dissertation.

[2] R. Penrose (1990). Twistor Theory for Vacuum Space-Times: a New Approach, T N **31**, 6–8.

Franz Müller

To appear in: P. Bergmann's Festschrift, ed. N. Sanchez:

Twistors as Spin 3/2 Charges

by Roger Penrose

Mathematical Institute

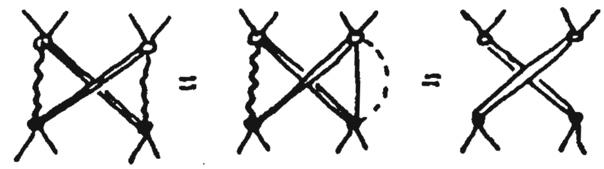
Oxford, UK

Abstract It is pointed out that twistors play a role as the charges for helicity 3/2 massless fields. Since such fields can be defined consistently in general Ricci-flat 4-manifolds, a possible new approach to defining twistors in vacuum

Towards Twistor Vacuum Diagrams

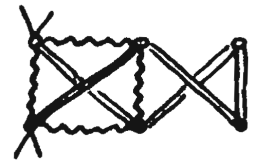
By using the Twistor skeleton expressions for first order Feynman diagrams and the idea of representing a propagator as a sum over states (APH TN31) it is easy to write down twistor expressions for low order Feynman vacuum diagrams. But it is not clear how to go about constructing the contours over which one integrates them.

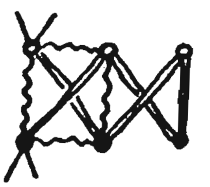
As a guide one knows in the Möller case one can take the period in one of the (inhomogeneous) boundaries to give a non-interaction diagram, which is equivalent to

$k_4 \frac{\partial}{\partial k_4}$ 

Thus in the sum-over-states case one needs a contour which will preserve the analogous version of this.

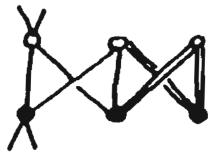
The integral one is interested in is: (all lines are inhomogeneous). One wants the extra boundary lines at infinity to be able to treat the internal double poles as restricting (A.P.H. this issue). The left side of this diagram is completed using one of the new Hodges spinor integrals (A.P.H. TN30). The inhomogeneous scale which is left is integrated from the boundary through the logarithmic cut and back to the boundary on the next sheet. Then using standard results from double-box diagrams gives

$I =$ 

 $= \frac{1}{\substack{AB \\ CD}} \left(1 + \log\left(\frac{k_4 k_3}{k_1 k_2}\right)\right)$.

On applying $k_4 \frac{\partial}{\partial k_4}$ one gets  as required.

Now one can use an analogous contour on:

$J =$ 

Which gives $J = 0$, and justifies taking J as the twistor translation of the φ^4 tadpole diagram  which is zero in the massless limit.

Thus a contour exists for the twistor translation of $L =$ , which gives $L = 0$. Work is in progress to extend this treatment to vacuum diagrams like  .

This work was done in conjunction with A. Hodges.

TWISTOR NEWSLETTER No. 32

Contents

Short contributions for TN 33 should be sent to

Thomas von Schroeter, Editor, Twistor Newsletter,
Mathematical Institute,
24-29 St. Giles,
Oxford OX1 3LB, England.
email: TVS@UK.AC.OX.VAX

to arrive before the 4th of October 1991.