Twistors as Spin 3/2 Charges Continued: $SL(3,\mathbb{C})$ Bundles

In TN 32 (and in a new Festschrift volume for P.G. Bergmann), I made the suggestion that a concept of twistor, appropriate for Ricci-flat space-times generally, would be as a (conserved) <u>charge</u> for massless helicity 3/2 fields. This suggestion is based on the following two observations:

① $R_{ab}=0$ provides the consistency condition (in an appropriate sense) for the existence and propagation of such field, in curved space-time $\mathcal{M}$ (Buchdahl, Deser & Zumino, Julia, Chinea(-Tod));

② In Minkowski space-time $\mathbb{M}$, the space of charges for such fields is naturally identified with the twistor space $\mathbb{T}$ of $\mathbb{M}$.

Thus, it would seem that if the appropriate concept of "charge" for massless helicity 3/2 fields in a Ricci-flat $\mathcal{M}$ could be found, then this should provide the long-sought concept of twistor appropriate to the vacuum Einstein equations. By analogy with the "nonlinear graviton construction", the long-term programme is thus as follows:

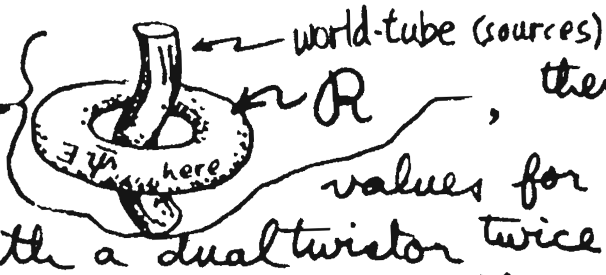
ⓐ Find this concept of twistor for Ricci-flat $\mathcal{M}$;

ⓑ Find out how to characterize the geometry of this resulting twistor space $\mathcal{T}$ — the hope being that the general such $\mathcal{T}$ can be constructed with free functions;

ⓒ Find how to reconstruct $\mathcal{M}$ from $\mathcal{T}$.

The concept of "charge" in ② above is closely analogous to the concept of energy-momentum/angular momentum that arises for linear gravity — the sources for a spin 2 massless fields in $\mathbb{M}$ (conserved). If we think of this linear gravity spin 2 field as being described by a (Weyl tensor-like) object

$$K_{abcd} = \Psi_{ABCD}\,\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{A}\varepsilon_{B}\,\overline{\Psi}_{A'B'C'D'},$$

where $\overline{\Psi}_{A'B'C'D'} = \overline{\Psi}_{(A'B'C'D')}$ satisfies the massless helicity 2 field equation $\nabla^{AA'}\overline{\Psi}_{A'B'C'D'}$ in some region $\mathcal{R}$ of $\mathbb{M}$ surrounding a world-tube → 

, then we can obtain the "charge" values for the tube by spin-lowering with a dual twistor twice ($2 \xrightarrow[\text{twistor}]{\text{dual}} \frac{3}{2} \xrightarrow[\text{twistor}]{\text{dual}} 1$) to obtain a self-dual Maxwell field, whose charges (electric + i× magnetic) can then be obtained by a Gauss integral, as is done in the quasi-local mass construction. Since two dual twistors are involved, the charges for linear gravity come out as the components of a symmetric valence 2 twistor $A^{\alpha\beta}$. The similar construction for a massless helicity 3/2 field $\psi_{A'B'C'}$ ($=\psi_{(A'B'C')}$; $\nabla^{AA'}\psi_{A'B'C'}=0$) involves only one dual twistor ($\frac{3}{2} \xrightarrow[\text{tw.}]{\text{dual}} 1$), so we get the charges as components of a valence 1 twistor, say Z^{α}.

For a general unrestricted $A^{\alpha\beta}$ ($=A^{(\alpha\beta)}$) there would be three types of component, with dependence upon position x^a:

quadratic in x^a — angular momentum (6 real cpts.) (primary part of $A^{\alpha\beta}$)

linear in x^a — $\begin{cases}\text{4-momentum (4 real cpts.)}\\ \text{"magnetic" (NUT) 4-mom. (4 real cpts.)}\end{cases}$

independent of x^a — "anti-angular momentum" (6 real cpts.) (projection part of $A^{\alpha\beta}$)

Total: 20 real cpts (= 10 complex cpts.)

But when there is an appropriate potential h_{ab} for $\psi_{A'B'C'D'}$ in $\mathcal{R}$ (with $\psi_{A'B'C'D'} = \frac{1}{2}\nabla^{A}_{(A'}\nabla^{B}_{B'}h_{C'D')AB}$; cf. S.&S.-T. vol. 1, p. 364), or, equivalently, when the sources in the world-tube come from a (linearized) energy-momentum tensor (cf. S.&S.-T. vol. 2, §6.2), we have $A^{\alpha\beta}I_{\beta\gamma} = \bar{A}_{\gamma\beta}I^{\beta\alpha}$; then the "anti-angular momentum" and "magnetic" (NUT) components vanish and the order of dependence on position for the remaining components is reduced by one: linear in x^a for angular momentum ; independent of x^a for 4-momentum.

All this is rather analogous to what occurs for spin $\frac{3}{2}$. Here we have

linear in x^a — ω-part of Z^α (primary part) 4 real cpts.
independent of x^a — π-part of Z^α (projection part) 4 real cpts

Total: (8 real cpts. =) 4 complex cpts.

Somewhat analogous to the condition $A^{\alpha\beta}I_{\beta\gamma} = \bar{A}_{\gamma\beta}I^{\beta\alpha}$, arising from the existence of h_{ab} globally in $\mathcal{R}$, is the condition $Z^\alpha I_{\alpha\beta} = 0$ (i.e. $\pi_{A'} = 0 \therefore \omega^A = \text{const.}$) arising from the existence of a first potential $\gamma^A_{B'C'}$ for $\psi_{A'B'C'}$ globally throughout $\mathcal{R}$. Here we can adopt either the Dirac form $\boxed{D}$ of the potential, for which the symmetry

$$\boxed{D} \qquad \gamma^A_{B'C'} = \gamma^A_{C'B'}$$

holds, or else the Rarita-Schwinger form $\boxed{R\text{-}S}$, for which this symmetry is not imposed. The equations to be satisfied are

$$\boxed{D} \qquad \nabla^{BB'}\gamma^A_{B'C'} = 0$$

or

$$\boxed{R\text{-}S} \quad \begin{cases} \nabla^{B'(B}\gamma^{A)}_{B'C'} = 0 & \cdots\cdots (1) \\ \varepsilon^{B'C'}\nabla_{A(A'}\gamma^A_{B')C'} = 0 & \cdots\cdots (2) \end{cases}$$

with gauge freedom $\gamma^A_{B'C'} \mapsto \gamma^A_{B'C'} + \nabla^A_{B'}\nu_{C'}$,
where

$$\boxed{D} \qquad \nabla^{AA'}\nu_{A'} = 0$$

(but $\nu_{A'}$ is free in the $\boxed{R\text{-}S}$ case). These equations also work in Ricci-flat $\mathcal{M}$.

We can also define the "field" $\psi_{A'B'C'}$ $(= \psi_{(A'B'C')})$ by

$$\boxed{D} \qquad \psi_{A'B'C'} = \nabla_{AA'}\gamma^A_{B'C'}$$

or

$$\boxed{R\text{-}S} \quad \psi_{A'B'C'} = \nabla_{A(A'}\gamma^A_{B')C'}.$$

In $\mathbb{M}$, $\psi_{A'B'C'}$ is also gauge invariant and satisfies the field equation $\nabla^{AA'}\psi_{A'B'C'}$. However, for this to be true in Ricci-flat $\mathcal{M}$, we also need the condition that the Weyl tensor is anti-self-dual (ASD; i.e. $\tilde{\Psi}_{A'B'C'D'} = 0$). In that case, we can define the "π-space" for twistors by

$$\mathbb{S}_{A'} = \{\text{glob. } \psi\text{s}\}/\{\text{glob. } \gamma\text{s}\}$$

where "{glob. $\hat{\psi}_s$}" means the space of fields $\psi_{A'B'C'}$ that are global throughout $\mathcal{R}$, and correspondingly for "{glob. γ_s}". This does not work when the Weyl tensor is not ASD, however, even though there is apparently a natural-looking suggestion for a meaning for "{glob. $\hat{\psi}_s$}" in terms of a patchwork of gauge-equivalent γ_s (see R.P. in TN 32). Taking an open covering $\{\mathcal{U}_i\}$ for $\mathcal{R}$, we can try to define $\overset{i}{\gamma}$ on $\mathcal{U}_i$ where $\overset{i}{\gamma}-\overset{j}{\gamma}$ is pure gauge on each $\mathcal{U}_i \cap \mathcal{U}_j$. Unfortunately we find, in general, that any such "global ψ" is equivalent to a global γ.

It is instructive to compare this with the spin 2 case where we can consider choosing $\overset{i}{h}_{ab}$ on $\mathcal{U}_i$, to represent a linearized perturbation of the metric, with

$$\overset{i}{h}_{ab} - \overset{j}{h}_{ab} = 2\nabla_{(a}\xi_{b)} \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j$$

(gauge equivalence). We find that for this to represent a genuine metric perturbation on the whole of $\mathcal{R}$, we must have, for every triple overlap

$$\overset{ij}{\xi} + \overset{jk}{\xi} + \overset{ki}{\xi} = 0 \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$$

(and also, in accordance with the phenomenon for γ_s, as described in TN 32, does not count as "gauge equivalence" between $\overset{1}{h}$ and $\overset{2}{h}$ unless $\overset{12}{\xi} = \overset{12}{\xi}'$ in the central region of the overlap}).

This condition on triple overlaps serves to exclude "magnetic mass" (i.e. a NUT-type source). It is analogous to ruling out "$\pi_{A'}$-charge" in the spin 3/2 case, which is just what we do not want to rule out. This suggests that we do something in the spin 3/2 case that is analogous to Misner's solution to the problem of eliminating the NUT ("Dirac-type") singular wires — the triple regions on which $\overset{12}{\xi} + \overset{23}{\xi} + \overset{31}{\xi} = 0$ fails. Misner noted that NUT space could be made non-singular ("wire-free") by means of identifications in the time direction that introduced closed timelike curves in the space-time. This is only possible because the NUT space-time has a timelike Killing vector, and the symmetry in that direction allows identifications to be made.

If we are to try something similar in the general spin $\frac{3}{2}$ case, we would want to erect some kind of bundle over $\mathcal{M}$ which will admit the necessary extra symmetries in the fibre direction. We also need γ to have some kind of non-linear interpretation (since we do not expect a linear π-space in the general case). The best bet appears to be to interpret γ as providing a <u>bundle connection</u>.

Take the fibre coordinates to be given by a spinor $\eta_{A'}$ and a scalar ζ. For a given small ϵ, we extend the ordinary (Christoffel-Levi-Civita) covariant derivative ∇ on $\mathcal{M}$ to bundle-valued quantities according to

$$\nabla_{PP'}\begin{pmatrix}\eta_{A'}\\ \zeta\end{pmatrix} = \begin{pmatrix}\nabla_{PP'}\eta_{A'}\\ \nabla_{PP'}\zeta\end{pmatrix} - \epsilon\begin{pmatrix}0 & \gamma_{PP'A'}\\ \gamma_{PP'}{}^{B'} & 0\end{pmatrix}\begin{pmatrix}\eta_{B'}\\ \zeta\end{pmatrix},$$

the gauge transformations being

$$\begin{pmatrix}\eta_{A'}\\ \zeta\end{pmatrix} \longmapsto \begin{pmatrix}\eta_{A'}\\ \zeta\end{pmatrix} + \epsilon\begin{pmatrix}0 & \nu_{A'}\\ \nu^{B'} & 0\end{pmatrix}\begin{pmatrix}\eta_{B'}\\ \zeta\end{pmatrix},$$

where terms of order ϵ^2 are being neglected. (There appears to be some relation to supersymmetry operations here, but as far as I can make out, what I am doing is not the same; also, an interesting limiting case of the above (studied by K.P.T.) occurs when one of the two off-diagonal entries in the matrices is set to zero, but this seems not to lead to the needed non-linearities.) As they stand, the gauge transformations do not close under commutation, and for a consistent theory valid to all orders in ϵ we need to generalize to $SL(3,\mathbb{C})$ matrices before the commutation closes. Writing $\textcircled{A}$ for the three-dimensional indices occurring here, so $\eta_{\textcircled{A}} = \begin{pmatrix}\eta_{A'}\\ \zeta\end{pmatrix}$, etc., we have a connection defined according to:

$$\nabla_{PP'}\eta_{\textcircled{A}} = \begin{pmatrix}\nabla_{PP'}\eta_{A'}\\ \nabla_{PP'}\zeta\end{pmatrix} - \gamma_{PP'\textcircled{A}}{}^{\textcircled{B}}\,\eta_{\textcircled{B}}$$

with gauge trans. $\eta_{\textcircled{A}} \mapsto \eta_{\textcircled{A}} + \nu_{\textcircled{A}}{}^{\textcircled{B}}\,\eta_{\textcircled{B}}$.

The $\nu_{\textcircled{A}}{}^{\textcircled{B}}$ are $SL(3,\mathbb{C})$-valued fields on $\mathcal{M}$, so

$$\epsilon^{\textcircled{P}\textcircled{Q}\textcircled{R}}\,\nu_{\textcircled{P}}{}^{\textcircled{A}}\,\nu_{\textcircled{Q}}{}^{\textcircled{B}}\,\nu_{\textcircled{R}}{}^{\textcircled{C}} = \epsilon^{\textcircled{A}\textcircled{B}\textcircled{C}},$$

where $\epsilon^{\textcircled{P}\textcircled{Q}\textcircled{R}}$ (and $\epsilon_{\textcircled{P}\textcircled{Q}\textcircled{R}}$) are Levi-Civita objects, so

$$\epsilon^{\textcircled{A}\textcircled{B}\textcircled{C}}\,\overset{1}{\eta}_{\textcircled{A}}\,\overset{2}{\eta}_{\textcircled{B}}\,\overset{3}{\eta}_{\textcircled{C}} = \overset{1}{\eta}_{A'}\overset{2}{\eta}{}^{A'}\overset{3}{\zeta} + \overset{2}{\eta}_{A'}\overset{3}{\eta}{}^{A'}\overset{1}{\zeta} + \overset{3}{\eta}_{A'}\overset{1}{\eta}{}^{A'}\overset{2}{\zeta}.$$

The γs are likewise generalized, and we have an object like

$$\gamma_{PP'Ⓐ}{}^{Ⓑ} = \begin{pmatrix} \alpha_{PP'A'}{}^{B'} & \beta_{PP'A'} \\ \gamma_{PP'}{}^{B'} & \delta_{PP'} \end{pmatrix}.$$

The curvature is

$$K_{pqⒶ}{}^{Ⓑ} = 2\nabla_{[p}\gamma_{q]Ⓐ}{}^{Ⓑ} + 2\gamma_{[p|Ⓐ|}{}^{Ⓒ}\gamma_{q]Ⓒ}{}^{Ⓑ}.$$

It seems that we must consider this as a generalization of the $\boxed{\text{R-S}}$ rather than the $\boxed{\text{D}}$ form, since the commutators of $\boxed{\text{D}}$ gauge quantities cease to satisfy $\nabla_A^{A'}\nu_{A'}=0$ or any other sensible local equation. One is led to speculate that an appropriate non-linear generalization of the $\boxed{\text{R-S}}$ equations might well be

$$^{-}K_{PQⒶ}{}^{Ⓑ} = 0, \quad ^{+}K_{P'Q'Ⓐ}{}^{Ⓑ}\epsilon^{P'ⒶⒸ}\epsilon_{Q'ⒷⒹ} = 0$$

(or some such), where

$$K_{pqⒶ}{}^{Ⓑ} = {}^{-}K_{PQⒶ}{}^{Ⓑ}\varepsilon_{P'Q'} + \varepsilon_{PQ}{}^{+}K_{P'Q'Ⓐ}{}^{Ⓑ}$$

is the splitting of the curvature into anti-self-dual and self-dual pieces respectively. (These eqns. are respective analogues of ① and ②.) Here, $\epsilon^{P'ⒶⒸ} = \epsilon^{ⓅⒶⒸ}e_{Ⓟ}{}^{P'}$, $\epsilon_{Q'ⒷⒹ} = \epsilon_{ⓆⒷⒹ}e_{Q'}{}^{Ⓠ}$, where $e_{Ⓟ}{}^{P'}$ and $e_{Q'}{}^{Ⓠ}$ are appropriate (degenerate) "soldering" quantities relating the bundle directions with tangent (spinor) directions in $\mathcal{M}$, and there are presumably also equations on these, like perhaps $\nabla_{PP'}e_{Ⓟ}{}^{P}=0$. This is very much "work in progress", and the appropriate geometrical viewpoint has not fully come to light. (There are some possible relations between this type of bundle and other constructions of relevance to twistor theory. If these become clearer they will be reported on later.)

Note that the equation on $^{-}K_{\dots}{}^{\bullet}$ tells us that we have a <u>self-dual connection</u> of a particular type. Thus, in the special case when $\mathcal{M}$ has a self-dual Weyl curvature, the Ward construction should lead to an interpretation of this connection in terms of $\mathcal{M}$'s <u>dual</u> twistor space, as if the charges could be defined for this connection, we have a different angle on the googly.

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