# The Initial Value Problem in General Relativity by Power Series.

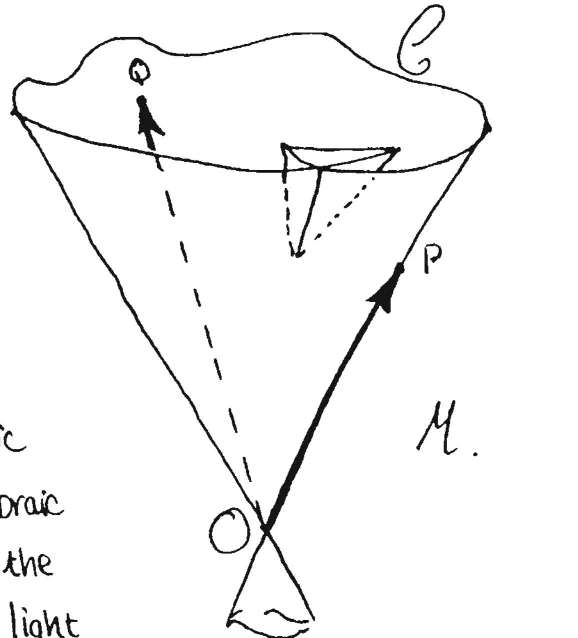
We work throughout in a curved vacuum space-time. At some point $O$, we take the light cone $\mathscr{C}$. Suppose we know the structure of $\mathscr{C}$, in the sense that we can express the component of the Weyl curvature spinor, $\Psi$, along any null ray. Assuming an analytic space-time we use a Taylor expansion, which when we take the component along the null ray gives, for any point $P$ on $\mathscr{C}$,

$$\Psi_{ABCD}(P)\xi^A\xi^B\xi^C\xi^D = \Psi_{ABCD}(O)\xi^A\xi^B\xi^C\xi^D + r\nabla_{AA'}\Psi_{BCDE}(O)\xi^A\bar{\xi}^{A'}\xi^B\xi^C\xi^D\xi^E$$

$$+\frac{r^2}{2!}\nabla_{AA'}\nabla_{BB'}\Psi_{CDEF}(O)\xi^A\bar{\xi}^{A'}\xi^B\bar{\xi}^{B'}\xi^C\xi^D\xi^E\xi^F + \dots$$

where $P$ is displaced by parameter value $r$ from $O$ along a null geodesic defined by the null vector $\xi^A\bar{\xi}^{A'}$ at $O$.

This form only requires the symmetric part of the derivatives of $\Psi$ to find the component of $\Psi$ at any point $P$ on $\mathscr{C}$.




So we have the geometry of $\mathscr{C}$ determined by the set of all symmetric derivatives of $\Psi$. Now, by similarly doing a Taylor expansion for the whole space-time $\mathcal{M}$, we can express the whole $\Psi$ at any point $Q$ in terms of derivatives (unsymmetrised) of $\Psi$ at $O$.

But we know that the symmetric derivatives of $\Psi$ form an exact set, an algebraic basis (Penrose + Rindler, 1984), so that knowing the elements of this set (i.e. the structure of the light cone) – $\{\Psi_{ABCD}, \Psi_{ABCDE}^{E'}, \Psi_{ABCDEF}^{E'F'}, \dots\}$ where $\Psi_{ABCDEF\dots H}^{E'F'\dots H'} = \nabla_{(E}^{(E'}\nabla_F^{F'}\dots\nabla_H^{H')}\Psi_{ABCD)}$ allows us to calculate the unsymmetrized derivatives of $\Psi$ and thus determine the structure inside the light cone. This was first suggested by Penrose in 1963 (rep. GRG 12).

We want therefore to find expressions which give the unsymmetrised derivatives of $\Psi$, completely in terms of those symmetrised derivatives of $\Psi$.

i.e. unsymmetrised $n^{th}$ derivative of $\Psi$ =

$\Psi_n$ + function in lower order $\Psi_n$'s.

where $\Psi_n$ represents the $n^{th}$ symmetrized derivative of $\Psi$.

The process of expressing the derivative of $\Psi$ in this way will involve substituting for $\Psi_n$'s of lower order which have already been calculated. The d'Alembertian operator on each of the $\Psi_n$ will also be needed.

The calculation is iterative in form. Starting with the last $\Psi_n$ calculated, a $\nabla$ is applied and the indices symmetrised. The symmetrisation is done so that the resulting expression has all the indices still in alphabetical order. This is the canonical form which is set for spinor expressions to allow them to be simplified to their lowest form.

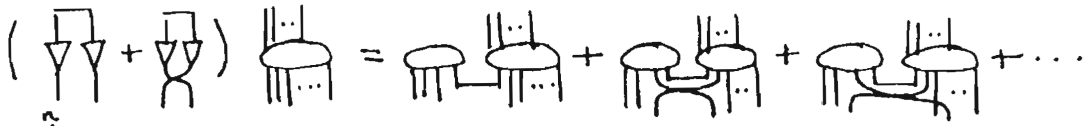
It is possible to simplify general spinor expressions by reducing them to canonical form, using the ε-identity to uncross indices.

This does produce a large number of terms. Also we need to eliminate those parts of an expression of the form $\alpha_A \alpha_B \varepsilon^{AB}$, which equal zero. This is done by symmetrising over $\alpha$'s indices.

$$\alpha_A \alpha_B \varepsilon^{AB} \rightarrow 1/2 \left( \alpha_A \alpha_B \varepsilon^{AB} + \alpha_B \alpha_A \varepsilon^{BA} \right)$$
$$\rightarrow 1/2 \left( \alpha_A \alpha_B \varepsilon^{AB} - \alpha_A \alpha_B \varepsilon^{AB} \right) \rightarrow 0$$

In practice however we are symmetrising over huge numbers of indices, giving large number of terms in the expression. In order to avoid having to do too much of these simplifications, all indices are kept in canonical form as the calculation proceeds.

Continuing from the symmetrised derivative of $\Psi_n$, we substitute for $\Psi_n$. This gives an expression containing terms of at least two derivatives of $\Psi_{n-1}$. Terms in this expression can be simplified using the Ricci Identities, where the sum of two appropriate terms of two derivatives is replaced with a product of $\Psi$'s, i.e.



where  $= \Psi_n$.

There will also be terms of the form  to be substituted for. This will use an expression for $\Box \Psi_{n-1}$ which will have been calculated at the same time as $\Psi_{n-1}$, where

$$\Box \Psi_{n-1} = \sum \text{ terms in } \Psi_{n-1} \text{ and lower order } \Psi\text{'s}.$$

The expression can thus be reduced to one containing only $\Psi_n$'s and one term, the unsymmetrised $n^{th}$ derivative of $\Psi_{ABCD}$. These methods are also applied in calculating $\Box \Psi_{n-1}$ as part of the iterative process.

Starting with $\Psi_{ABCD}$ and knowing that $\varepsilon^{AB} \nabla_{AA'} \Psi_{BCDE} = 0$, we can easily calculate $\Psi_1 = \nabla_{AA'} \Psi_{BCDE}$ and can calculate $\Box \Psi_{ABCD}$ in terms of $\Psi_{ABCD}$. This is then take to calculate $\Psi_2$ in terms of $\Psi_1, \Psi_0$ plus $\nabla_{AA'} \nabla_{BB'} \Psi_{CDEF}$, and $\Box \Psi_1$ in terms of $\Psi_1$ and $\Psi_0$, and so on.

This process has been implemented in Mathematica. The expressions obtained are extremely long and Mathematica encounters certain difficulties in managing expressions of this length effectively. Explorations are being made to transfer this work to another system.

A more specialised approach to the problem would be to set the light cone to be converging, by giving specific values for the symmetric derivatives which set the structure of $\ell$. This can be achieved by,

$$\Psi_{ABCD} = a\, \iota^A \iota^B \iota^C \iota^D \quad \text{and} \quad \Psi^{E'}{}_{ABCDE} = b\, o^{E'} o_E\, o_A o_B o_C o_D$$

and all others are zero.

This reconverging light cone structure would mean that there must be a singularity in the space-time. The structure of the space-time as the singularity is approached could be investigated using this method.

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