Conformal Singularities and the Weyl Curvature Hypothesis

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Abstract *A conjecture of K.P. Tod relating Penrose's Weyl Curvature Hypothesis and the isotropy of the Universe has recently been proved. Background material and the method of proof are discussed.*

The large scale structure of the Universe is well-known to exhibit a high degree of spatial isotropy. Claims have been made that this may be due to quantum processes occurring at times subsequent to the Big Bang, such as inflation and dissipative particle interactions. According to Penrose (1977) however, one should seek a more fundamental explanation in terms of entropy considerations at the Big Bang itself. This view has recently been provided mathematical substantiation by a development in classical general relativity.

Although Penrose is concerned to apply a condition of zero entropy at the Big Bang, it is worthwhile to first recall a contrary view of Misner (1968) that the Big Bang was maximally disordered. One might seek to motivate this philosophy of 'chaotic cosmology' on the grounds that physical laws based on field equations are inherently inapplicable at singularities, and that the Big Bang, being singular, cannot therefore be constrained by such laws. As will be seen, this view may be naive. But whatever its motivation, the major problem for chaotic cosmology is to explain how the high degree of large-scale isotropy of the present-day Universe could have evolved from an chaotic Big Bang. Neutrino-induced viscosity (Matzner and Misner (1972)) and curvature-induced creation of particle pairs (Zel'dovich (1972)) are just two of the possible isotropization mechanisms that have been investigated. However further studies (Collins and Stewart (1971), Collins and Hawking (1973), Barrow and Matzner (1977)) indicate that that the age and expansion rate of the Universe place such severe constraints on the effects of such processes that gravitational

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instabilities would have been likely to predominate, causing the growth of any initial anisotropies. Chaotic cosmology therefore appears to be untenable:

Chaos then implies chaos now.

Consider then the alternative view that the Big Bang was ordered. One might reasonably hope that such order would be found to have its origins in the laws of physics rather than be a manifestation of initial conditions imposed at the will of a Creator. But in any case, the principal problem for 'quiescent cosmology' (Barrow (1978)) is to show that an appropriately ordered Big Bang necessarily evolves to a spatially isotropic Universe. That is:

Order then implies order now.

This problem breaks naturally into two parts. First, to determine a suitable condition of order at the Big Bang. And second, to establish the uniqueness of the evolution despite the singular nature of the initial state. The problem, as a whole, was given a mathematically clear formulation in the context of classical general relativity by Tod (1987). His fundamental proposal was that the Big Bang be assumed to be a conformal singularity, meaning that it could be conformally transformed into a smooth spacelike hypersurface. With regard to the condition of order at the Big Bang, he adopted Penrose's Weyl Curvature Hypothesis (Penrose (1977), (1981), (1986)) that the Weyl tensor tends to zero at the Big Bang. This Hypothesis is motivated by a speculation that the Weyl tensor may be related to an, as yet undefined, gravitational entropy which is initially zero, but increases as gravitational instabilities, and ultimately gravitational collapse, come into effect. This in turn is motivated firstly by calculations which show that singularities resulting from gravitational collapse have a tendency towards high anisotropy, and correspondingly large Weyl tensor, and secondly by a need (Penrose (1979)) to account for the temporal assymmetry of the Universe. But to return to the problem in hand, Tod imposed a final condition that the Universe be assumed to be filled by an isentropic perfect fluid. His conjecture then was that, subject to the stated conditions, the Universe is necessarily spatially isotropic and therefore Robertson-Walker.

The study of conformal singularities arose out of a well-known series of papers on singularities in general relativity by Lifschitz and co-workers, beginning with Lifschitz and Khalatnikov (1963) and culminating with Belinski et al. (1982). Unfortunately, in view of their use of power series approximations, not all their conclusions can be considered well-founded. Nonetheless, their work did indicate that, in modern terminology, the induced conformal 3-metric on the initial hypersurface representing a conformal singularity determines the 4-dimensional Riemann tensor there. Subsequent calculations of Goode and Wainwright (1985) (who employed the term 'isotropic singularity') enabled Tod (1987) to conclude that the Weyl Curvature Hypothesis implies that the conformal 3-metric on the initial hypersurface is of constant curvature, and therefore isotropic and likely to give rise to a Robertson-Walker Universe. Certainly any constant curvature 3-metric can arise as the initial conformal 3-metric for some Robertson-Walker model. The outstanding problem was therefore to show that the initial conformal 3-metric constitutes sufficient initial data to uniquely determine the subsequent evolution. This has now been carried out for the special case $\gamma = 4/3$ corresponding to a Universe filled by radiation or a highly relativistic fluid. (The adiabatic index γ, in general a function of the fluid density ρ, determines the pressure p of the fluid according to $p = (\gamma - 1)\rho$.)

Theorem (Newman (1991). A $\gamma = 4/3$ perfect fluid space-time which evolves from a spacelike conformal singularity subject to the Weyl Curvature Hypothesis is necessarily Robertson-Walker near the singularity. ■

This result gives an affirmative resolution of the Tod conjecture in the case $\gamma = 4/3$. It is likely that a routine extension of the techniques involved in the proof may permit more general equations of state, although asymptotic conditions on γ for large matter density ρ will undoubtedly be necessary.

One technical point should be made. Previous authors, and Goode and Wainwright (1985) in particular, permitted or even implicitly demanded the conformal factor in the description of the conformal singularity to be non-differentiable at the initial hypersurface representing the singularity. This gave rise to considerable complications. For present purposes however, the conformal factor is assumed, along with the conformal manifold and the conformal metric, to be C^∞. This has the curious, but desirable consequence that, as the singularity is approached, γ must tend to the value $4/3$ appropriate to a hot Big Bang. For mathematical simplicity the theorem assumes that γ has what is therefore its only possible constant value in this context, namely $\gamma = 4/3$.

Although the proof of the theorem is long, the underlying method is sufficiently straightforward to be outlined here. The first task is to fix the conformal factor. It can be shown (Scott (1988)) that the velocity of the fluid is necessarily irrotational in both the physical and conformal pictures, and meets the initial hypersurface orthogonally in the latter. One can therefore demand that the conformal factor be such that its level surfaces are orthogonal to the fluid velocity. The conservation equations then suggest a natural scaling in relation to the fluid density ρ . Following Goode and Wainwright (1985) one now shows that, within the conformal picture, the magnetic part of the Weyl tensor must vanish at the initial hypersurface, whilst the electic part is proportional to the trace-free part of the Ricci tensor of the conformal 3-metric thereon. The Weyl Curvature Hypothesis and the contracted Bianchi identities are now sufficient to show that this conformal 3-metric is of constant curvature.

The irrotationality of the fluid velocity suggests the use of comoving coordinates. However it is well known that, at least for the vacuum Einstein equations, hyperbolicity and the consequent well-posedness of the Cauchy problem are most easily demonstrated in harmonic coordinates. Nonetheless, comoving coordinates turn out to be the better choice. Independent variables are now selected in such a manner as to obtain from the conformally transformed Einstein equations for the fluid a first order quasi-linear symmetric hyperbolic system of evolution equations of the form:

$$A^0(u)\,\partial_t u = A^i(u)\,\partial_i u + (B(u) + t^{-1}C(u))u\,.$$

$$u = \overset{\bullet}{u} \quad \text{at} \quad t = 0$$

Here u is a 63-component column vector, $A^0(u), \ldots , A^3(u)$, $B(u)$ and $C(u)$ are 63×63 matrices, analytic in u , with $A^0(u), \ldots , A^3(u)$ symmetric (hence the 'symmetry' of the system) and $A^0(u)$ positive definite (hence the 'hyperbolicity'). Also $(A^0(u))^{-1}C(u)$ has no positive integer eigenvalues. The quantity t is the conformal factor which also serves as a time coordinate. The spatial coordinates are labelled by $i = 1, 2, 3$. The function $\overset{\bullet}{u}$ on the initial hypersurface is fixed by the conformal 3-metric there together with the gauge conditions. One can show that the system not only follows from, but is equivalent to the original conformally transformed Einstein equations from which it was derived. However this is not relevant to the Tod conjecture for which it would suffice to work with any of a number of smaller systems that are not known to posess this property.

In order to complete the proof, it remains to establish a uniqueness theorem for solutions to equations of the above form. Somewhat surprisingly, no suitable theorem was to be found in the literature, so a special study had to be undertaken. The property that $(A^0(u))^{-1}C(u)$ has no positive integer eigenvalues plays a fundamental role. To see this, suppose for simplicity that one wishes to establish uniqueness only amongst analytic solutions. For any such solution u, the above equation may be differentiated $n-1$ times and restricted to $t=0$ to yield

$$(\mathrm{Id} - n^{-1}(A^0(0))^{-1}C(0))\, \partial_t^n|_{t=0}u \; = \; \text{terms in } \partial_t^p|_{t=0}u \text{ and } \partial_t^p|_{t=0}\partial_i u$$

$$i = 1, 2, 3\,; \quad 0 \le p \le n-1\,; \quad n \ge 1$$

$$u = \overset{\bullet}{u} \quad \text{at} \quad t = 0$$

For each positive integer n one can thus, by induction, express $\partial_t^n|_{t=0}u$ in terms of $\partial_i^p \overset{\bullet}{u}$, $0 \le p \le n-1$, $i = 1, 2, 3$. It follows that u, being analytic, is uniquely determined on a neighbourhood of the initial hypersurface $t = 0$ in terms of $\overset{\bullet}{u}$ and its derivatives. One can in fact not only dispense with analyticity, but work at finite levels of differentiability by means of fixed point techniques. Even at the C^∞ level some subtlety is required because the coefficient t^{-1} in the basic system of equations tends to resist contraction mappings. The difficulties can be overcome however to yield the required uniqueness theorem and hence a proof of the Tod conjecture in the case $\gamma = 4/3$.

An outstanding problem is to show that the conformal 3-metric on the initial hypersurface constitutes a complete set of initial data, even without the imposition of the Weyl Curvature Hypothesis. To have obtained a symmetric hyperbolic system equivalent to the original conformally transformed Einstein equations is a significant advance. A suitable existence theorem for this system would complete the result.

The work described here concerning perfect fluid space-times in the vicinity of conformal singularities in many ways parallels work of Friedrich (1985) concerning vacuum space-times in the vicinity of conformal infinity. Symmetric hyperbolic systems form the key ingredient in both instances, although in the case of conformal infinity one does not have to contend with a t^{-1} forcing term. Perhaps a unified treatment may be possible. But in any case it is a remarkable feature of the Einstein equations, with or without matter, that the fundamental property of hyperbolicity can survive conformal transformations, even where the conformal factor passes through zero or infinity. Whilst the mathematics of this phenomenon is part way to being understood, the underlying physical significance remains mysterious.

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