Twistor theory and integrability

This note consists mostly of certain speculative and conjectural comments that I gave or would have liked to give in my 5 minute contribution to the special twistor workshop to celebrate the birthday of the founder of many of these ideas.

In this note I wish to emphasize that the recently established deep links between twistor theory and integrable systems should lead to new techniques and results in twistor theory as well as unification and hopefully new results in the theory of integrable systems.

In various articles it has emerged that many/most integrable systems are symmetry reductions of the self-dual Yang-Mills equations with a small number of exceptions, most notable of which is the KP hierarchy. Furthermore, much of the theory and structure of these equations can be understood in a reasonably direct way as various features of the symmetry reductions of the Ward correspondence for the self-dual Yang-Mills equations. See Mason & Sparling and references therein.

As far as the equations are concerned, it appears that we can classify most integrable systems as reductions from self-dual Yang-Mills in 4-dimensions by choice of:

a) a gauge group,

b) a symmetry group (with a possible discrete component),

c) a normal form for the various constants of integration that arise in the reduced equation.

For example, the Drinfeld Sokolov systems can all be understood in this way as can various other large classes of integrable systems.

The standard theory of the equations consists of such constructions as the inverse scattering transform, actions of loop groups and realizations of the P.D.E.s as flows on grassmanians. These can be understood as various ansätze and normal forms for the patching data of the holomorphic vector bundles on twistor space with the appropriate symmetry properties that arise from the corresponding symmetry reductions of the Ward transform for the self-dual Yang-Mills equations.

However, these ideas from the theory of integrable systems are in many cases refinements of twistor ideas, and in others are completely new in twistor theory. There is therefore the possibility of methods from the theory of integrable systems being used to solve problems in twistor theory. The following conjectures and connections include examples where twistor theory may benefit from this interaction.

1) Inverse scattering. The inverse scattering transform provides a parametrization of the solution space of integrable P.D.E.s. The parameters can be used to build patching data for the Ward bundle on twistor space directly. For example for the attractive nonlinear Schrodinger equation we get the solution space identified with: $Map(S^1 \mapsto D) \times \amalg_{k=1}^{\infty} S^k\{C^* \times D\}$ where D is the unit disc in the complex plane C, C^* are the non-zero comlex numbers, $\amalg$ is the disjoint union and S^k is the symmetrized cartesian product. The first factor are solutions that one would expect from linearizing the equations (for which they are the Fourier transform) but the second factor are the soliton solutions which do not have a linear analogue.

One would expect this pattern to be generic for solutions of the self-dual Yang-Mills equations in indefinite signature. So one would expect for example that on the compactified 4-dimensional Minkowski space with signature (2,2) the solution space of the $SU(n)$ self-dual Yang-Mills equations is a Cartesian product of maps from RP^3 to unit determinant Hermitean $n \times n$ matrices with a soliton type sector, which would presumably be the (2,2) analogues of instantons. It is perhaps worth mentioning that the first factor can be understood as a nonlinear generalization of the Radon transform. A similar picture should hold for the symmetry reductions to equations in $2+1$ dimensions and other $1+1$ dimensional systems.

2) The inverse scattering transform in 2+1 dimensions such as for the KP hierarchy has features that distinguish it clearly from existing twistor correspondences so that there seems little real hope of incorporating it into the above framework. Nevertheless, it is a natural generalization of the framework for the KdV equations and leads to a coherent inverse scattering transform based on a non-local Riemann-Hilbert transform. One may hope, then, that the transform can be articulated geometrically so that it leads to some new category of twistor constructions.

It is perhaps worth remarking that the pseudo-differential operators that play such a prominent role in the KP equations also arose naturally in one of RP's earlier discussions of the googly problem—the patching operation was represented by a pseudo-differential operator representable by integration against a kernel just as in the KP inverse scattering problem.

Another point is that the inverse scattering transform does work for many other field equations in higher dimensions but is no longer implementable by linear procedures and hence does not lead to practical solution generation methods. It may nevertheless lead to a workable framework for understanding general relativity using spin 3/2 fields and RP's elemental states based on asymptotic twistors (see the previous TN).

3) It is possible to use the Ward correspondence to understand the connections between the KdV type equations and the 2-dimensional quantum field theory of free Fermions, developed by the Japanese school and described in Segal & Wilson and Witten. Solutions (at least those that are reflectionless) of the KdV equations are given by amplitudes associated to flows acting on certain special vectors in the free Fermion Fock space. The link is that the free Fermions are the holomorphic sections of the Ward bundle on twistor space restricted to a complex projective line, and the quantum field theoretic amplitude in question is the 2-point function that gives rise to the Greens function for the $\bar{\partial}$-operator. Finding the Greens function is equivalent to trivializing the vector bundle on the line which is the key step in obtaining the self-dual Yang-Mills field in terms of the bundle.

One may ask the question then of whether its possible to realize other more complicated twistor constructions such as the nonlinear graviton construction as a more complicated, perhaps interacting 2-dimensional quantum field theory. In particular this might explain the remarkable link discovered by Ooguri & Vafa between $N = 2$ string theory and the self-dual Einstein equations.

4) There is much scope for using ideas from the quantum inverse scattering transform to understand how to use twistor methods in the context of integrable quantum field theory. In particular the Russian school's introduction of the R-matrix to describe the Poisson bracket structure should pass over directly to give the Poisson bracket relations for the twistor patching data. Other workers have managed to show that the inverse scattering transform survives quantization so that one can hope to quantize on twistor space and then transform the results to obtain a quantum field theory on space-time. The existing theory is still in need of further insights that twistor theory may be able to provide.

5) Witten has attempted a unification of the theory of integrable statistical mechanical models using Chern-Simons quantum field theory. This produces R-matrices, and is sufficient for understanding knot polynomials. Unfortunately it does not provide the dependence of the R-matrices on the spectral parameter that is so crucial to integrability. So it is not possible to regard this as a satisfactory understanding of integrable statistical mechanics. One may conjecture that by studying a quantum field theory of self-dual Yang-Mills reduced to 3-dimensions this gap would be remedied.

It is perhaps also worth drawing attention to the Atiyah-Murray conjecture also in this context (see their article in the last TN).

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