The 3-Wave Interaction from the Self-dual Yang Mills Equations

There is a family of completely integrable systems called 'the n-wave interaction' (see eg Ablowitz and Segur 1981). According to current twistor dogma, these equations should be reductions of the self-dual Yang-Mills equations. While trying to do something different, I found a way of getting them by this route. I also found, though somewhat later, that Chakravarty and Ablowitz (1990) had a slightly different route with similar end-points.

The starting point is the self-dual Yang-Mills equations with 2 null symmetries. These are equivalent to the commutation relation

\[ [D_1, D_2] = 0 \tag{1} \]

where

\[ D_1 = \delta_1 - A_1 + \zeta B_1, \]
\[ D_2 = \delta_2 - A_2 + \zeta B_2, \tag{2} \]

the \( A_i \) and \( B_i \) are \( n \times n \) complex matrices, functions of \( x^1 \) and \( x^2 \) only, and \( \zeta \) is a complex constant.

Substituting (2) into (1) and equating separate powers of \( \zeta \) to zero gives 3 equations. The \( O(\zeta^2) \) term is just

\[ [B_1, B_2] = 0 \tag{3} \]

Mason and Singer (1991; see also Mason 1991) solve this by taking the \( B_i \) to be nilpotent and arrive at the \( n \)-th generalised KdV equation. The opposite extreme, which I shall take, is to suppose that the \( B_i \) are diagonalisable by the Yang-Mills gauge freedom, which is

\[ B_i \to G^{-1} B_i G; \quad A_i \to G^{-1} (A_i G - \delta_i G). \tag{4} \]

where \( G \) is an \( n \times n \) complex matrix depending on \( x^1 \) and \( x^2 \).

Now the \( O(\zeta) \) term in (1) is

\[ \delta_1 B_2 - \delta_2 B_1 + A_2 B_1 + B_2 A_1 - A_1 B_2 - B_1 A_2 = 0 \tag{5} \]

The diagonal entries in (5) imply that there is a 'potential' for the \( B_i \):

\[ B_i = \delta_i C \tag{6} \]

while the off-diagonal entries imply that the off-diagonal entries of the \( A_i \) are proportional in a way that I shall write out explicitly below.

Before that, we consider the \( O(1) \) term in (1) which is

\[ \delta_2 A_1 - \delta_1 A_2 + A_1 A_2 - A_2 A_1 = 0 \tag{7} \]

The diagonal entries in (7) imply that the diagonal entries of the \( A_i \) have potentials in a way analogous to (6). A gauge transformation (4) with diagonal \( G \) preserves the diagonality of the \( B_i \) and can be chosen to remove the diagonal entries of the \( A_i \).
To summarise the situation at this point in the argument:

(i) the $B_i$ are diagonal and derived from a potential $C$ as in (6);
(ii) the $A_i$ are purely off-diagonal and can be expressed in terms of $C$ and each other using (5);
(iii) finally (7) imposes some differential equations.

At what is essentially this point, Chakravarty and Ablowitz (1990) take the matrices $B_i$ to be constant and arrive at the $n$-wave interaction. This is a specialisation in that the $B$'s can't in general be made constant by a gauge transformation (4), but it leads to the same equations eventually as we shall see.

Now it is necessary to resort to taking components so for simplicity I will restrict to $3 \times 3$ matrices. Set

$$B_1 = \text{diag}(\alpha, \beta, \gamma) = \delta_1 C; \quad B_2 = \text{diag}(\lambda, \mu, \nu) = \delta_2 C \quad (8)$$

and

$$\alpha - \beta = \delta_1 P \quad \beta - \gamma = \delta_1 Q \quad \gamma - \alpha = \delta_1 R$$
$$\lambda - \mu = \delta_2 P \quad \mu - \nu = \delta_2 Q \quad \nu - \lambda = \delta_2 R \quad (9)$$

so that

$$P + Q + R = 0. \quad (10)$$

We will eventually switch to using two of $P, Q, R$ as independent variables.

With the choices (8) for the $B_i$, we can solve (5) for the $A_i$ in terms of another off-diagonal matrix $E$. Set

$$A_1 = (a_{1i}) \quad A_2 = (b_{1i}) \quad E = (e_{1i})$$

then (5) implies

$$a_{12} = (\alpha - \beta)e_{12}; \quad b_{12} = (\lambda - \mu)e_{12} \quad (11)$$

and the 5 equations obtained from this by the obvious permutations.

Finally, we substitute (11) into (7) to obtain differential equations on $E$. These differential equations can all be written with the aid of the Poisson bracket in $(x', x^2)$. The typical one, from which the other 5 follow by permutations, is

$$(E_{12}, R) = E_{13}E_{22}(P, Q) \quad (12)$$

We can break the symmetry between $P, Q, R$ by adopting $P$ and $Q$ as new independent coordinates. Write 'dot' and 'prime' for differentiation w.r.t $P$ and $Q$ respectively and set

$$E = \begin{pmatrix}
0 & H & V \\
W & 0 & F \\
G & U & 0
\end{pmatrix}$$

then (12) becomes the system
\[ P' = -UV \quad U' = GH \quad (13) \]
\[ \dot{G} = WU \quad \dot{V} = -FH \]
\[ \dot{H} - H' = -UV \quad \dot{W} - W' = FG \]

which is equivalent to the 3-wave interaction.

The further reduction 'dot = minus prime' leads, after some manipulating of constants, to the integrable Hamiltonian

\[ h = p_1 p_2 q_2 + q_1 q_2 p_2 \]

References


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