The 3-Wave Interaction from the Self-dual Yang Mills Equations

There is a family of completely integrable systems called 'the n-wave interaction' (see eg Ablowitz and Segur 1981). According to current twistor dogma, these equations should be reductions of the self-dual Yang-Mills equations. While trying to do something different, I found a way of getting them by this route. I also found, though somewhat later, that Chakravarty and Ablowitz (1990) had a slightly different route with similar end-points.

The starting point is the self-dual Yang-Mills equations with 2 null symmetries. These are equivalent to the commutation relation

$$[D_1, D_2] = 0 \qquad (1)$$

where

$$D_1 = \partial_1 - A_1 + \zeta B_1$$
$$D_2 = \partial_2 - A_2 + \zeta B_2, \qquad (2)$$

the A_i and B_i are n×n complex matrices, functions of x^1 and x^2 only, and ζ is a complex constant.

Substituting (2) into (1) and equating separate powers of ζ to zero gives 3 equations. The $O(\zeta^2)$ term is just

$$[B_1, B_2] = 0 \qquad (3)$$

Mason and Singer (1991; see also Mason 1991) solve this by taking the B_i to be nilpotent and arrive at the n-th generalised KdV equation. The opposite extreme, which I shall take, is to suppose that the B_i are diagonalisable by the Yang-Mills gauge freedom, which is

$$B_i \to G^{-1} B_i G \ ; \ A_i \to G^{-1}(A_i G - \partial_i G). \qquad (4)$$

where G is an n×n complex matrix depending on x^1 and x^2.

Now the $O(\zeta)$ term in (1) is

$$\partial_1 B_2 - \partial_2 B_1 + A_2 B_1 + B_2 A_1 - A_1 B_2 - B_1 A_2 = 0 \qquad (5)$$

The diagonal entries in (5) imply that there is a 'potential' for the B_i:

$$B_i = \partial_i C \qquad (6)$$

while the off-diagonal entries imply that the off-diagonal entries of the A_i are proportional in a way that I shall write out explicitly below. Before that, we consider the O(1) term in (1) which is

$$\partial_2 A_1 - \partial_1 A_2 + A_1 A_2 - A_2 A_1 = 0 \qquad (7)$$

The diagonal entries in (7) imply that the diagonal entries of the A_i have potentials in a way analogous to (6). A gauge transformation (4) with diagonal G preserves the diagonality of the B_i and can be chosen to remove the diagonal entries of the A_i.

To summarise the situation at this point in the argument:

(i) the B_i are diagonal and derived from a potential C as in (6);
(ii) the A_i are purely off-diagonal and can be expressed in terms of C and each other using (5);
(iii) finally (7) imposes some differential equations.

At what is essentially this point, Chakravarty and Ablowitz (1990) take the matrices B_i to be constant and arrive at the n-wave interaction. This is a specialisation in that the B's can't in general be made constant by a gauge transformation (4), but it leads to the same equations eventually as we shall see.

Now it is necessary to resort to taking components so for simplicity I will restrict to 3×3 matrices. Set

$$B_1 = \mathrm{diag}(\alpha,\beta,\gamma) = \partial_1 C \; ; \qquad B_2 = \mathrm{diag}(\lambda,\mu,\nu) = \partial_2 C \qquad (8)$$

and
$$\alpha - \beta = \partial_1 P \qquad \beta - \gamma = \partial_1 Q \qquad \gamma - \alpha = \partial_1 R \qquad (9)$$
$$\lambda - \mu = \partial_2 P \qquad \mu - \nu = \partial_2 Q \qquad \nu - \lambda = \partial_2 R$$

so that
$$P + Q + R = 0. \qquad (10)$$

We will eventually switch to using two of P,Q,R as independent variables.

With the choices (8) for the B_i, we can solve (5) for the A_i in terms of another off-diagonal matrix E. Set

$$A_1 = (a_{ij}) \quad A_2 = (b_{ij}) \quad E = (e_{ij})$$

then (5) implies

$$a_{12} = (\alpha-\beta)e_{12} \; ; \; b_{12} = (\lambda-\mu)e_{12} \qquad (11)$$

and the 5 equations obtained from this by the obvious permutations.

Finally, we substitute (11) into (7) to obtain differential equations on E. These differential equations can all be written with the aid of the Poisson bracket in (x^1,x^2). The typical one, from which the other 5 follow by permutations, is

$$\{E_{12},R\} = E_{13}E_{32}\{P,Q\} \qquad (12)$$

We can break the symmetry between P,Q,R by adopting P and Q as new independent coordinates. Write 'dot' and 'prime' for differentiation w.r.t P and Q respectively and set

$$E = \begin{pmatrix} 0 & H & V \\ W & 0 & F \\ G & U & 0 \end{pmatrix}$$

then (12) becomes the system

$$\begin{aligned} F' &= -VW & U' &= GH \\ \dot{G} &= WU & \dot{V} &= -FH \\ \dot{H} - H' &= -UV & \dot{W} - W' &= FG \end{aligned} \qquad (13)$$

which is equivalent to the 3-wave interaction.

The further reduction 'dot = minus prime' leads, after some manipulating of constants, to the integrable Hamiltonian

$$h = p_1 p_2 q_3 + q_1 q_2 p_3$$

References

M.J.Ablowitz and H.Segur 1981 *Solitons and the Inverse Scattering Transform* SIAM Philadelphia

S.Chakravarty and M.J.Ablowitz 1990 *On reductions of self-dual Yang-Mills equations* University of Colorado preprint PAM 62

L.J.Mason 1991 in **TN30**

L.J.Mason and M.A.Singer 1991 private communication (preprint in preparation)

Paul Tod