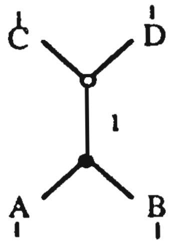
Twistor diagrams in higher dimensions

The single box in higher dimensions

The observation that integration of homogeneous twistor-differential forms (twistor diagrams) over chosen $\mathbb{Z}$-cycles [C], such as for example

$$\int_{\mathbb{C}^4\times\mathbb{C}^{4*}\supset C_1,\,C_2}$$



$$= \pm \begin{cases} \left(\dfrac{A\cdot B}{C\cdot D}\right)^{-1} \lg \left(\dfrac{A\,B}{C\,D} \Big/ \dfrac{A\,B}{D\,C} \right) & (C_1) \\[2ex] \left(\dfrac{A\cdot B}{C\cdot D}\right)^{-1} & (C_2) \end{cases} \qquad (1)$$

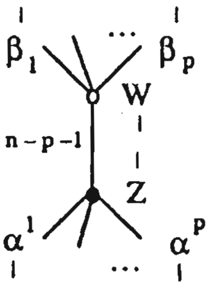
result in expressions which make sense in any dimension (e.g.

$$A,\, B,\, C,\, D \in \mathbb{C}^{n\,(*)}, \quad n \geq 2\ ) \qquad (2)$$

raises the natural question:

Are there higher (or in some cases also lower) dimensional forms and cycles (contours) which reproduce those same results?

In pursuit of this question – which will also have interesting implications for the inversion of differential operators corresponding to massless propagators – we start with the following procedure of "inflating" a particular contour $C_{n,p}$ along the internal line of a twistor diagram on $\mathbb{C}^n{}_Z \times \mathbb{C}^{n*}{}_W$:



$$\longleftrightarrow \quad \frac{(n-p-1)!\, D^n Z\, D^n W}{\alpha_1\cdot Z \,\cdots\, \alpha_p\cdot Z\, (W\cdot Z)^{n-p}\, W\cdot\beta_1 \,\cdots\, W\cdot\beta_p} =: \omega_n \qquad (3)$$

$$D^n Z = dZ^1 \ldots dZ^n / n! \quad , \quad \det\begin{pmatrix}\alpha^1 \ldots \alpha^p \\ \beta_1 \ldots \beta_p\end{pmatrix} \neq 0 \quad , \quad p < n \quad .$$

We extend the exterior parameters to bases

$$\{\alpha^1,\ldots,\alpha^n\},\ \{\beta_1,\ldots,\beta_n\} \subset (\mathbb{C}^n_Z)^*,\ \mathbb{C}^n_W \quad \text{with} \quad \alpha^i\cdot\beta_j = \delta^i_j \quad \text{for } \max\{i,j\} > p\,;$$

with their duals

$$\{\alpha_1,\ldots,\alpha_n\,(=\beta_n)\} \subset \mathbb{C}^n_Z\ (\equiv \mathbb{C}^n_W),\ \{\beta^1,\ldots,\beta^n\,(=\alpha^n)\} \subset \mathbb{C}^{n*}_W\ (\equiv (\mathbb{C}^n_Z)^*)$$

so that

$$\overset{|}{Z} = \sum_{i=1}^{n} \overset{\alpha^i\,|}{\underset{Z}{|}}\,\alpha_i = \Sigma\, z^i \overset{|}{\alpha}_i \quad , \quad \underset{|}{W} = \Sigma \overset{W}{\underset{\beta_i\,|}{|}}\,\beta^i = \Sigma\, w_i \underset{|}{\beta}^i \quad . \tag{4}$$

Integration of ω_n over the standard cycle $[C_{n,p}] = [(S^1)^{2p+1} \times S^{2n-2p-1}]$ with

$$(S^1)^{2p+1} \times \underbrace{S^{2n-2p-1}}_{S^1 \times_{Hopf} \mathbb{C}P^{n-p-1}} = \left\{ \begin{matrix} ((z^1,\ldots,z^n), \\ (w_1,\ldots,w_n)) \end{matrix} \;\middle|\; \begin{matrix} z^j = \varepsilon e^{i\phi_j},\, w_j = \varepsilon e^{i\psi_j},\ j \le p\,; \\ w_j = \bar{z}_j e^{i\phi},\, \Sigma z^j \bar{z}_j = 1,\, j > p\,; \\ \phi\,,\phi_j,\,\psi_j \in [0, 2\pi],\ \varepsilon \text{ const} \ll 1 \end{matrix} \right\} \tag{5}$$

gives

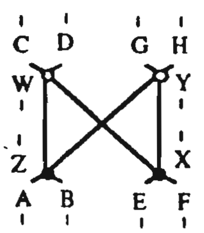
$$\int_{C_{n,p}} \omega_n = (2\pi\, i)^{n+p+1} \left(\frac{\overset{\alpha^1}{\scriptstyle ||} \ \cdots\ \overset{\alpha^p}{\scriptstyle |}}{\underset{\beta_1}{\scriptstyle ||} \ \cdots\ \underset{\beta_p}{\scriptstyle |}} \right)^{-1} \quad . \tag{6}$$

The contour $C_{n,p}$ and the differential form ω_n can obviously be constructed for any $n > p$ (and indeed also for $n = p$). In the following commuting embeddings we regard $C_{m,p}$ for $m \ge n$ as the "inflated" version of the contour $C_{n,p}$:

$$\begin{array}{ccc} C_{n,p} & \hookrightarrow & C_{m,p} \\ \downarrow & & \downarrow \\ \mathbb{C}^n \times \mathbb{C}^{n*} & \hookrightarrow & \mathbb{C}^m \times \mathbb{C}^{m*} \end{array} \tag{7}$$

To get contours B_n for higher dimensional box diagrams we apply this procedure twice, first to the base and then to the fibres over this "inflated" base of a standard contour B_0 for the box diagram in

$\mathbb{C}^4{}_Z \times \mathbb{C}^{4*}{}_W \times \mathbb{C}^4{}_X \times \mathbb{C}^{4*}{}_Y$:



$$\longleftrightarrow \quad \omega_{0,0} = \frac{D^4 Z\, D^4 W\, D^4 X\, D^4 Y}{\begin{matrix} \text{YABWWWWEFYYY} \\ |\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,| \\ \text{ZZZZ C D XXXXGH} \end{matrix}} \quad , \quad \underbrace{\frac{\overset{AB}{||}}{\underset{CD}{||}}}_{=:\, a} \underbrace{\frac{\overset{EF}{||}}{\underset{GH}{||}}}_{} \underbrace{\frac{\overset{AB}{||}}{\underset{GH}{||}}}_{} \underbrace{\frac{\overset{EF}{||}}{\underset{CD}{||}}}_{=:\, c} \neq 0 \tag{8}$$

If the ingoing and outgoing states are not too far apart

$$\frac{\overset{AB}{||}}{||} \approx \frac{\overset{EF}{||}}{||} \quad , \quad \frac{||}{\underset{GH}{||}} \approx \frac{||}{\underset{CD}{||}} \tag{9}$$

B_0 can be described as a fibre bundle

$$\begin{array}{ccccccccc} (S^1)^8 & \sim & C_{4,3} & \sim & F_0 & \longrightarrow & B_0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \pi_0 B_0 & \sim \; C_{4,2} \; \sim & (S^1)^5 \times S^3 \end{array} \tag{10}$$

where π_0 is the projection onto the factor $\mathbb{C}^4{}_X \times \mathbb{C}^{4*}{}_Y$. The base space is constructed to be a $C_{4,2}$ as in (5) using bases as in (4)

$$\{\underset{|}{\alpha}^1,\ldots,\underset{|}{\alpha}^4\},\ \{\overset{|}{\beta}_1,\ldots,\overset{|}{\beta}_4\} = \{\underset{|}{E},\underset{|}{F},\underset{|}{\alpha}^3,\underset{|}{\alpha}^4\},\ \{\overset{|}{G},\overset{|}{H},\overset{|}{\beta}_3,\overset{|}{\beta}_4\};\ \frac{\alpha^i}{\beta_j} = \delta^i_j \quad \text{for } \max\{i,j\} > 2\,; \tag{11}$$

with the additional diagonalising property

$$\frac{\underset{|}{A}\underset{|}{B}\underset{|}{\alpha}^i}{\overset{|}{C}\overset{|}{D}\overset{|}{\beta}_j} = \frac{\underset{|}{A}\underset{|}{B}}{\overset{|}{C}\overset{|}{D}}\,\mu(i)\,\delta^i_j \quad \text{for } i,j > 2. \tag{12}$$

(9) then implies that $\mu(3), \mu(4) \approx 1$ and hence

$$\left|\frac{\underset{|}{A}\underset{|}{B}\underset{|}{X}}{\overset{|}{C}\overset{|}{D}\overset{|}{Y}} \Big/ \frac{\underset{|}{A}\underset{|}{B}}{\overset{|}{C}\overset{|}{D}}\right| \approx 1 + O(\varepsilon) \neq 0 \qquad \text{for all } (\overset{|}{X}, \underset{|}{Y}) \in \pi_0 B_0\,. \tag{13}$$

This allows us to fix the fibres $F_{0\,(\overset{|}{X},\underset{|}{Y})}$ as standard $C_{4,3}$'s ($\sim (S^1)^8$) by choices of bases as in (4)

$$\{\underset{|}{\gamma}^1,\ldots,\underset{|}{\gamma}^4\},\ \{\overset{|}{\delta}_1,\ldots,\overset{|}{\delta}_4\} = \{\underset{|}{A},\underset{|}{B},\underset{|}{Y},\underset{|}{\gamma}^4\},\ \{\overset{|}{C},\overset{|}{D},\overset{|}{X},\overset{|}{\delta}_4\};\ \frac{\gamma^i}{\delta_j} = \delta^i_j \quad \text{for } \max\{i,j\} > 3\,; \tag{14}$$

and thus B_0 is completely defined.

Given the trivial embedding

$$\mathbb{C}^4_Z \times \mathbb{C}^{4*}_W \times \mathbb{C}^4_X \times \mathbb{C}^{4*}_Y \hookrightarrow \mathbb{C}^{4+n}_Z \times \mathbb{C}^{4+n*}_W \times \mathbb{C}^{4+n}_X \times \mathbb{C}^{4+n*}_Y \,. \tag{15}$$

we now construct a contour $B_n \hookleftarrow B_0$ in this enlarged space. We first define an "inflated" base $\pi_n B_n \sim C_{4+n,2}$ in the manner of (5) by extending the bases (11) subject to the same (extended) condition (12) where we can arrange $\mu(>4) = 1$. Then (13) continues to be satisfied for all $(\overset{|}{X},\underset{|}{Y}) \in \pi_n B_n$ and, by extensions of (14), we can equally define "inflated" fibres $F_{n\,(\overset{|}{X},\underset{|}{Y})}$ over $\pi_n B_n$ with

$$F_{0\,(\overset{|}{X},\underset{|}{Y})} \sim C_{4,3} \hookrightarrow F_{n\,(\overset{|}{X},\underset{|}{Y})} \sim C_{4+n,3} \quad \left((\overset{|}{X},\underset{|}{Y}) \in \pi_0 B_0 \hookrightarrow \pi_n B_n\right) \tag{16}$$

If (9) holds exactly one has

$$\left\langle \underset{|}{\alpha}^{3},\dots,\underset{|}{\alpha}^{n+4}\right\rangle = \left\langle \frac{\overset{|\,|\,|}{ABY}}{\underset{|\,|\,|}{CD}} = \frac{\overset{|\,|\,|}{EFY}}{\underset{|\,|\,|}{GH}}, \underset{|}{\gamma}^{3},\dots,\underset{|}{\gamma}^{n+4}\right\rangle \quad \text{and}$$

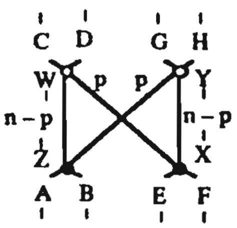
$$\left\langle \overset{|}{\beta}_{3},\dots,\overset{|}{\beta}_{n+4}\right\rangle = \left\langle \frac{\overset{|\,|}{AB}}{\underset{|\,|\,|}{CDX}} = \frac{\overset{|\,|}{EF}}{\underset{|\,|\,|}{GHX}}, \overset{|}{\delta}_{4},\dots,\overset{|}{\delta}_{n+4}\right\rangle \qquad (17)$$

and a restriction to extensions which are orthonormal in the sense

$$\frac{\gamma^{i}}{\bar{\gamma}_{j}} = \delta^{i}_{j} = \frac{\bar{\delta}^{i}}{\delta_{j}}\ , \quad \text{where} \quad \overline{z\underset{|}{\alpha}^{i}} = \bar{z}\overset{|}{\beta}_{i} \quad \text{for} \quad z \in \mathbb{C}\ ;\ i,j = 3,\dots,n+4,$$

defines $F_n(\overset{|}{X},\underset{|}{Y})$ uniquely. The general case with (9) follows from continuity.

Thus we get a contour B_n over which we can integrate the forms



$$\longleftrightarrow \quad \omega_{n-p,p} = \frac{[(n-p)!\,p!]^{2}\, D^{4+n}Z\, D^{4+n}W\, D^{4+n}X\, D^{4+n}Y}{\begin{pmatrix} Y \\ | \\ Z\end{pmatrix}^{p+1} \frac{AB}{ZZ}\begin{pmatrix} W \\ | \\ Z\end{pmatrix}^{n-p+1} \frac{WW}{CD}\begin{pmatrix} W \\ | \\ X\end{pmatrix}^{p+1} \frac{EF}{XX}\begin{pmatrix} Y \\ | \\ X\end{pmatrix}^{n-p+1} \frac{YY}{GH}}\ . \qquad (18)$$

Integrating over the fibres $F_n(\overset{|}{X},\underset{|}{Y})$ we get

$$\int_{B_n} \omega_{n-p,p} = (2\pi i)^{n+8}\, p!\,(n-p)! \left(-\frac{\overset{|\,|}{AB}}{\underset{|\,|}{CD}}\right)^{p} \int\!\!\int_{\pi_n B_n} \left[\left(\frac{\overset{|\,|\,|}{ABY}}{\underset{|\,|\,|}{CDX}}\right)^{p+1} \begin{pmatrix} Y \\ | \\ X\end{pmatrix}^{n-p+1} \frac{EFYY}{XXGH}\right]^{-1} D^{4+n}X\, D^{4+n}Y$$

and after integrating out $(S^1)^4$ around the exterior poles $\Sigma_i E_i X^i$, $F_i X^i$, $Y_i G^i$, $Y_i H^i = 0$ we are left with

$$(2\pi i)^{n+12}(-1)^{p}\, p!\,(n-p)!\, a^{-1} \int_{S^1 \times S^{3+2n}} \bigwedge_i dx^i \bigwedge_j dy_j \left[\sum_k \mu(k)\, x^k y_k\right]^{-p-1} \left[\sum_h x^h y_h\right]^{p-1-n}$$

$$(2 < i,j,k,h \le n+4\ ,\ \mu(>4) = 1)$$

$$\overset{\text{if } n>0}{=} (2\pi i)^{2n+15}\, \frac{(-1)^{p}\, p!\,(n-p)!}{n!}\, n\, a^{-1} \int_0^1 dt \int_0^1 \frac{(1-s)^{n-1}\, s\, ds}{(1+\kappa(t)s)^{p+1}}\ , \quad \kappa(t) = \mu(3)t + \mu(4)(1-t) - 1$$

(19)

and one finds that

$$\frac{1}{(2\pi i)^{2n+15}} \int_{B_n} \sum_{p=0}^{n} (-1)^{p} \binom{n}{p} \omega_{n-p,p} = \frac{\lg \lambda_1/\lambda_2}{\lambda_1 - \lambda_2} \quad \text{independent of } n \ge 0! \qquad (20)$$

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where $\lambda_i = a\,\mu(2+i)$, $i = 1,2$, are the roots of

$$\lambda^2 - \Big(a + c - \begin{matrix} A\,B\,E\,F \\ |\;|\;|\;| \\ C\,D\,G\,H \end{matrix}\Big)\lambda + ac = 0\,. \tag{21}$$

The significance of this result lies in the fact that the left hand side of (20) can be written as

$$\frac{1}{(2\pi i)^{2n+15}} \int_{B_n} \Big[(\overline{ZX}\,\underline{\partial_Z\partial_X} \,/\, \overline{ZX}\,\underline{WY})^n\, K\Big]\, F\; DZ\, DW\, DX\, DY \tag{22}$$

with $\quad K^{-1} = \begin{matrix} W\,W\,Y\,Y \\ |\;|\;|\;| \\ Z\,X\,Z\,X \end{matrix}\,, \quad F^{-1} = \begin{matrix} A\,B\,E\,F\,W\,W\,Y\,Y \\ |\;|\;|\;|\;|\;|\;|\;| \\ Z\,Z\,X\,X\,C\,D\,G\,H \end{matrix}$

$\sqcup$, ($\sqcap$) is a fixed (hyper-)plane of complex (co-)dimension 2, generalising the infinity twistor. Thus if in a general scattering amplitude F contains a timelike propagator

$$(k_1 \cdot k_2)^{-1} \;\leftrightarrow\; (\overline{ZX}\,\underline{\partial_Z\partial_X})^{-1}\,, \quad F = (\overline{ZX}\,\underline{\partial_Z\partial_X})^{-1}\, \overline{ZX}\,\underline{WY}\,\tilde{F} \tag{23}$$

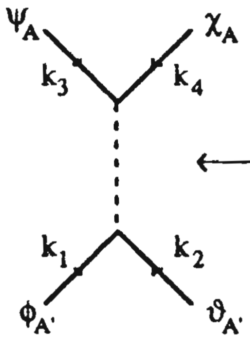
then integration by parts leads to the expression

$$\frac{1}{(2\pi i)^{2n+15}} \int_{B_n} \Big[(\overline{ZX}\,\underline{\partial_Z\partial_X} \,/\, \overline{ZX}\,\underline{WY})^{n-1}\, K\Big]\, \tilde{F}\,, \quad n > 0 \tag{24}$$

which in special examples can again be shown to be essentially independent of dimension.

<u>Scattering amplitudes</u>

As an example we consider the massless Yukawa process



$$\longleftrightarrow \int \Big[d^4k_1\, d^4k_2\, d^4k_3\, d^4k_4\, \delta(k_1 + k_2 + k_3 + k_4)\, \delta^-(k_4^2)\, \chi^A(k_4) \times \delta^-(k_3^2)\, \psi_A(k_3) \frac{1}{(k_1 + k_2)^2}\, \delta^+(k_2^2)\, \vartheta_{A'}(k_2)\, \delta^+(k_1^2)\, \phi^{A'}(k_1)\Big] \tag{25}$$

If we take special twistor representations for the exterior fields

$$\phi_{A'}(k_1) \leftrightarrow \overline{Z}\begin{pmatrix} A \\ | \\ Z \end{pmatrix}^{-2} \begin{pmatrix} B \\ | \\ Z \end{pmatrix}^{-1} \qquad \psi_A(k_3) \leftrightarrow \underline{W}\begin{pmatrix} W \\ | \\ C \end{pmatrix}^{-2} \begin{pmatrix} W \\ | \\ D \end{pmatrix}^{-1}$$

$$\vartheta_{A'}(k_2) \leftrightarrow \overline{X}\begin{pmatrix} E \\ | \\ X \end{pmatrix}^{-2} \begin{pmatrix} F \\ | \\ X \end{pmatrix}^{-1} \qquad \chi_A(k_4) \leftrightarrow \underline{Y}\begin{pmatrix} Y \\ | \\ G \end{pmatrix}^{-2} \begin{pmatrix} Y \\ | \\ H \end{pmatrix}^{-1} \tag{26}$$

where "$\leftrightarrow$" is the usual Penrose correspondence with respect to some fixed $\mathbb{C}P^{3(*)} \subset \mathbb{C}P^{3+n(*)}$, then

$$F = (\overline{ZX}\,\underline{\partial_Z\partial_X})^{-1}\, \overline{ZX}\,\underline{WY}\,\tilde{F} = (\overline{ZX}\,\underline{\partial_Z\partial_X})^{-1}\, \overline{ZX}\,\underline{WY} \begin{pmatrix} A\,E\,W\,Y \\ |\;|\;|\;| \\ Z\,X\,C\,G \end{pmatrix}^{-2} \begin{pmatrix} B\,F\,W\,Y \\ |\;|\;|\;| \\ Z\,X\,D\,H \end{pmatrix}^{-1} \tag{27}$$

and we can give the scattering amplitude in terms of higher dimensional single box twistor diagrams as

$$\frac{1}{(2\pi i)^{2n+15}}\int_{B_n}\left[(\partial_Z\partial_X/\underbrace{WY})^{n-1}K\right]\begin{pmatrix}A&E&W&Y\\ |&|&|&|\\ Z&X&C&G\end{pmatrix}^{-2}\begin{pmatrix}B&F&W&Y\\ |&|&|&|\\ Z&X&D&H\end{pmatrix}^{-1} D^{4+n}Z\,D^{4+n}W\,D^{4+n}X\,D^{4+n}Y \tag{28}$$

This is in fact the same as

$$\frac{1}{n(n+1)}\begin{bmatrix}\partial_C\partial_G & & \partial_C\partial_G\\ |\;\;| & - & |\;\;|\\ \partial_A\partial_B & & \partial_B\partial_A\end{bmatrix}\frac{1}{(2\pi i)^{15+2n}}\int_{B_n}\left[(\partial_Z\partial_X/\underbrace{WY})^{n}K\right]\frac{D^{4+n}Z\,D^{4+n}W\,D^{4+n}X\,D^{4+n}Y}{\begin{matrix}A\,E\,W\,Y\,B\,F\,W\,Y\\ |\;|\;|\;|\;|\;|\;|\;|\\ Z\,X\,C\,G\,Z\,X\,D\,H\end{matrix}} \tag{29}$$

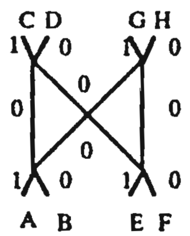
which can be calculated from (20)

$$\frac{1}{n(n+1)}\begin{bmatrix}\partial_C\partial_G & & \partial_C\partial_G\\ |\;\;| & - & |\;\;|\\ \partial_A\partial_B & & \partial_B\partial_A\end{bmatrix}\frac{\lg\lambda_1/\lambda_2}{\lambda_1-\lambda_2}=\left[\frac{\partial^2}{\partial\begin{smallmatrix}A\\|\\C\end{smallmatrix}\partial\begin{smallmatrix}B\\|\\G\end{smallmatrix}}-\frac{\partial^2}{\partial\begin{smallmatrix}B\\|\\C\end{smallmatrix}\partial\begin{smallmatrix}A\\|\\G\end{smallmatrix}}\right]\frac{\lg\lambda_1/\lambda_2}{\lambda_1-\lambda_2} \tag{30}$$

where on the right λ_1, λ_2 are taken to be functions of

$$z^1,\ldots,z^{16} = \begin{smallmatrix}A\\|\\C\end{smallmatrix},\begin{smallmatrix}A\\|\\D\end{smallmatrix},\begin{smallmatrix}A\\|\\G\end{smallmatrix},\begin{smallmatrix}A\\|\\H\end{smallmatrix},\begin{smallmatrix}B\\|\\C\end{smallmatrix},\begin{smallmatrix}B\\|\\D\end{smallmatrix},\begin{smallmatrix}B\\|\\G\end{smallmatrix},\begin{smallmatrix}B\\|\\H\end{smallmatrix},\begin{smallmatrix}E\\|\\C\end{smallmatrix},\begin{smallmatrix}E\\|\\D\end{smallmatrix},\begin{smallmatrix}E\\|\\G\end{smallmatrix},\begin{smallmatrix}E\\|\\H\end{smallmatrix},\begin{smallmatrix}F\\|\\C\end{smallmatrix},\begin{smallmatrix}F\\|\\D\end{smallmatrix},\begin{smallmatrix}F\\|\\G\end{smallmatrix},\begin{smallmatrix}F\\|\\H\end{smallmatrix}. \tag{31}$$

Thus we get an expression which is independent of the dimension n. It can be taken to define the left hand side in the case n = 0 which is of the form 0/0. For a twistor diagrammatic representation we take n=1:



(32)

<u>Remarks:</u>

1. There are also contours for the higher dimensional double box which allow representations of the right hand side of (20) involving space-like propagators and thus lead for example to a (dimension independent?) regularisation of Møller scattering.

2. It seems that a more invariant description of the contour B_n can be given as a bundle over the Grassmannian $Gr_2(\mathbb{C}^{2+n})$ with fibre B_0.

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