The Quantum Potential – Possible Origins?

If we substitute $\psi = Re^{iS/\hbar}$ into Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

and define $\rho = R^2$, $\underline{v} = \text{grad}\, S/m$, then we obtain the equations

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \underline{v}) = 0 \qquad ①$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}.\nabla \underline{v} = -\text{grad}\left(\frac{V}{m} - \frac{\hbar^2}{2m^2}\frac{\nabla^2 R}{R}\right) \qquad ②$$

Because of the definition of $\underline{v}$ we should also append

$$\text{curl}\, \underline{v} = 0 \qquad ③$$

[Using ②, this need only be imposed as an initial value constraint.]

The <u>quantum potential</u> is the term in ② given by

$$Q(x,t) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \qquad ④$$

If Q is omitted, then ① & ② describe the motion of an ensemble of Newtonian particles with velocity field $\underline{v}(x,t)$, and particle density $\rho(x,t)$ moving in potential field V. The continuity equation ① guarantees particle conservation and ② is the equation of motion

$$\frac{d^2\underline{x}}{dt^2} = -\text{grad}\,(\text{potential}/m) \quad \text{for each particle path } \underline{x}(t).$$

One is therefore led to a "classical" interpretation of Schrödingers equation in terms of an ensemble of classical particles moving in an <u>extra</u> potential given by ④, and all quantum effects are attributable to this peculiar term. These ideas are the starting point for De Broglie's pilot wave theory and Bohm's causal interpretation of quantum mechanics. It is interesting to speculate on the origin of $Q(x,t)$ with its unusually high order and nonlinear form.

We give here three, essentially mathematical, remarks, in the hope that they may lead to some physical insights.

① Solving the Continuity Equation.

The continuity equation ① is unchanged by external potentials and may be viewed as a "kinematical" relation (rather like "$v = \frac{dx}{dt}$" in particle mechanics, thinking of ρ as a kind of "position" variable). We can write down the general solution of ① and substitute it into ②.

One way to do this is to write

$$j^\mu = (\rho, \rho \underline{v}) \quad \mu = 0,1,2,3 \quad \text{so ① is } \partial_\mu j^\mu = 0$$

Then the general solution may be written

$$j^\mu = \epsilon^{\mu\alpha\beta\gamma} \partial_\alpha \lambda_1 \partial_\beta \lambda_2 \partial_\gamma \lambda_3$$

where $\lambda_1(x,t)$, $\lambda_2(x,t)$, $\lambda_3(x,t)$ are arbitrary functions.

This gives

$$\rho = \det \Lambda \quad , \quad v_i = -[\Lambda^{-1}]_{ki} \frac{\partial \lambda_k}{\partial t} \quad , \text{ where } \Lambda_{ij} = \partial_i \lambda_j \qquad ⑤$$

The λ's have the following geometrical interpretation:

At $t=0$ choose coordinates $\tilde{\underline{x}} = \varphi(\underline{x})$ making the density 1. This requires φ chosen so that

$$\rho(x,0) = \mathrm{Jac}(\partial \varphi / \partial \underline{x})$$

Then set $\underline{\lambda}(\underline{x},0) = \underline{\varphi}(\underline{x})$ and evolve constantly along the integral curves of $\underline{v}$ to get $\lambda(x,t)$.

i.e. $\lambda(x,t) = \tilde{x}$ coordinate at $t=0$ of the particle at (x,t).

Substituting the λ's into ② gives some rather complicated terms, but a remarkable simplification occurs:

In one space dimension, $\rho = \lambda_x \quad , \quad v = -\lambda_t / \lambda_x$

and we get :

$$\frac{\partial v}{\partial t} + v\cdot\nabla v = \left[-\frac{\lambda_{tt}}{\lambda_x} + \frac{2\lambda_t \lambda_{xt}}{\lambda_x^2} - \frac{\lambda_{xx}\lambda_t^2}{\lambda_x^3} \right] \qquad (6)$$

$$\mathrm{grad}\left(\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right) = \frac{-\hbar^2}{4m}\left[-\frac{\lambda_{xxxx}}{\lambda_x} + \frac{2\lambda_{xx}\lambda_{xxx}}{\lambda_x^2} - \frac{\lambda_{xx}^3}{\lambda_x^3} \right] \qquad (7)$$

Thus the quantum force term (7) arises by replacing t derivatives by xx derivatives in the convective derivative expression (6) ! This curious "symmetry" is also seen in the rather simple Lagrangian which generates all the terms in (6) and (7) :

$$L(\lambda_t, \lambda_x, \lambda_{xx}) = \frac{1}{2\lambda_x}\left[\lambda_t^2 - \left(\frac{\hbar^2}{4m^2}\right)\lambda_{xx}^2 \right] \qquad (8)$$

In the 3-dimensional case, where the terms are far more complicated (and (3) becomes nontrivial) the "symmetry" still holds, if a slightly different form of the general solution of the continuity equation is used. (to be described elsewhere)

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(2) Curvature Interpretations.

The expression (4) for the quantum potential is similar to various curvature formulas in differential geometry.

$2\nabla^2 R/R$ is the scalar curvature $\mathcal{R}$ of the 4-metric

$$ds^2 = \rho\, dt^2 - (dx^2 + dy^2 + dz^2)$$

and the free Schrödinger equation becomes

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = \mathrm{grad}\left(\frac{\hbar^2}{2m^2}\mathcal{R}\right) \qquad (9)$$

In a speculative mind, we may note that on the RHS, grad (scalar curvature) = div (2 Ricci tensor) [spatial components] and that the LHS of (9) is closely related to div $(T^{\mu\nu})$

for $T^{\mu\nu} = \rho v^\mu v^\nu$, $v^\mu = (1, \underline{v})$, the stress energy tensor of a perfect fluid. This suggests that we might consider removing the div's from both sides to obtain an underlying tensorial equation, rather similar to Einsteins equations!

Santamato (in Phys. Rev. 1984) has related the quantum potential to the curvature of a Weyl geometry in 3 dimensions.

Consider the Weyl connection ∇ characterised by the 1form Q:

$$\nabla_c g^{ab} = Q_c g^{ab}$$

In 3-dimensions take $g_{ab} = \delta_{ab}$, $Q_a = 2\partial_a \ln \rho$

Then the scalar curvature $R_{ab} g^{ab}$ of the Weyl connection gives $\nabla^2 R/R$. Note that Q is exact here, so the Weyl connection is equivalent to the metric connection of a conformally rescaled g_{ab} $(= \rho^2 \delta_{ab})$

3) a Gauge Theory?

In Schrödingers first paper on wave mechanics (Ann. Phys. 1926) he obtains his equation from the Hamilton-Jacobi equation

$$H\left(q, \frac{\partial S}{\partial q}\right) = E \qquad p = \frac{\partial S}{\partial q}$$

by the replacement $S \rightsquigarrow k \ln \psi$, $k = i\hbar$.

This replacement (which generates the quantum potential term in the ρ-$\underline{v}$ equation) may be written as (with $R, S, \underline{v}$ as before)

$$\frac{i m \underline{v}}{\hbar} \rightsquigarrow \frac{i m \underline{v}}{\hbar} + R^{-1} \operatorname{grad} R \qquad (10)$$

which is similar to the gauge potential transformation law

$$A \rightsquigarrow g^{-1} A g + g^{-1} \partial_\mu g \qquad (11)$$

(for an Abelian group). If we now introduce the $\underset{\sim}{\lambda}$-variables of (5), then (10) reads

$$\underline{v} \rightsquigarrow \underline{v} - \frac{i\hbar}{2m} \,\text{grad} \ln \det \Lambda \qquad (12)$$

The special form of the last term suggests we use the facts:

1. $\text{grad} \ln \det A = \text{tr}\,(A^{-1} \text{grad}\, A)$
2. $\text{tr}\,(B^{-1} A B) = \text{tr}\, A$
3. Using (5), v_i can be naturally written as a trace of a 3×3 matrix $V^{(i)}$

$$v_i = \text{tr}\,(V^{(i)}) \qquad V^{(i)}_{jk} = [\Lambda^{-1}]_{ji} \frac{\partial \lambda_k}{\partial t}$$

Then (12) becomes

$$\text{tr}\,(V^{(i)}) \rightsquigarrow \text{tr}\,(\Lambda^{-1} V^{(i)} \Lambda) - \frac{i\hbar}{2m} \text{tr}\,(\Lambda^{-1} \text{grad}\, \Lambda) \qquad (13)$$

Again we may imagine removing all the traces (and comparing to (11)) we obtain the gauge transformation law for a larger underlying non abelian theory.

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