An inverse twistor function

We know from the exact sequence argument that, under very general conditions, a zero-rest-mass field on complex Minkowski space can be generated from a twistor function.[1] There are various ways of constructing the twistor function explicitly. The first was found by Brian Bramson, George Sparling, and Roger Penrose.[2] In this note, I shall describe a very simple construction that extends Richard Ward's treatment of the positive homogeneity case.[3] It must be 'known', since it is really no more than a concrete version of the exact sequence argument, but I have not been able to find it in print, so it may not be 'well-known'.

I shall deal only with the wave equation, leaving the general case as an easy exercise. Let $\phi(x^a)$ be a holomorphic solution of $\Box\phi = 0$ on complex Minkowski space and let $o^{A'}$, $\iota^{A'}$ be a constant spinor dyad, with $o_{A'}\iota^{A'} = 1$. Put

$$f(x^a, \pi_{A'}) = \frac{\phi(x^a)}{o^{A'}\pi_{A'}\iota^{B'}\pi_{B'}}.$$

This function is defined on the primed-spinor bundle, less the zero sets of $o^{A'}\pi_{A'}$ and $\iota^{A'}\pi_{A'}$. It is homogeneous of degree -2 in $\pi_{A'}$, and it generates ϕ in the sense that

$$\phi(x^a) = \frac{1}{2\pi \mathrm{i}}\oint f(x^a, \pi_{A'})\pi_{B'}\mathrm{d}\pi^{B'}.$$

The only problem is that it does not descend to twistor space since it is not constant along the vector fields $\pi^{A'}\nabla_{AA'}$, which span the fibration of the primed-spinor bundle over $\mathbf{T}$. In fact,

$$\pi^{A'}\nabla_{AA'}f = \frac{\pi^{A'}\nabla_{AA'}\phi}{o^{B'}\pi_{B'}\iota^{C'}\pi_{C'}} = -\frac{\iota^{A'}\nabla_{AA'}\phi}{\iota^{B'}\pi_{B'}} + \frac{o^{A'}\nabla_{AA'}\phi}{o^{B'}\pi_{B'}}.$$

Let us look for holomorphic functions g_0 and g_1 on the complements in the primed-spinor bundle of $o^{A'}\pi_{A'} = 0$ and $\iota^{A'}\pi_{A'} = 0$, respectively, such that

$$\pi^{A'}\nabla_{AA'}g_0 = \frac{o^{A'}\nabla_{AA'}\phi}{o^{B'}\pi_{B'}} \quad \text{and} \quad \pi^{A'}\nabla_{AA'}g_1 = \frac{\iota^{A'}\nabla_{AA'}\phi}{\iota^{B'}\pi_{B'}}.$$

Then $f - g_0 + g_1$ will also generate ϕ, and will be constant along the fibration, and so will be a twistor function for ϕ.

The integrability conditions for the existence of g_0 and g_1 are satisfied as a consequence of the wave equation. In fact, if we can find a surface S_1 in complex Minkowski space that intersects each α-plane in the domain of ϕ in a single point, with the exception of the α-planes tangent to $\iota_{A'}$, then we can take

$$g_1(x^a, \pi_{A'}) = \int_x^y \frac{\iota^{A'}\iota_{B'}\nabla_{AA'}\phi\, \mathrm{d}x^{AB'}}{(\iota^{C'}\pi_{C'})^2},$$

where y is the intersection point of S_1 with the α-plane through x tangent to $\pi_{A'}$ and the integral is along any path in this α-plane from x to y. With $\pi_{A'}$ fixed, the integrand is closed, so the choice of path is immaterial (under the obvious topological conditions). We can similarly define g_0 by first choosing S_0 to intersect the α-planes that are not tangent to $o_{A'}$. (The requirements on S_0 and S_1 have been stated in a more restrictive form than is necessary: all we need is that g_0 and g_1 should be defined when $\pi_{A'}$ lies in appropriate neighbourhoods of $\iota_{A'}$ and $o_{A'}$, respectively.)

One possibility is to take S_0 to be a fixed α-plane parallel to $o_{A'}$ and S_1 to be a fixed α-plane parallel to $\iota_{A'}$. Then by taking x to lie on S_0, we can construct an explicit twistor function F as follows: let Z be an α-plane with tangent spinor $\pi_{A'}$ such that $o^{A'}\pi_{A'} \neq 0 \neq \iota^{A'}\pi_{A'}$ and put

$$F(Z) = \frac{\phi(x)}{o^{A'}\pi_{A'}\iota^{B'}\pi_{B'}} + \int_x^y \frac{\iota^{A'}\iota_{B'}\nabla_{AA'}\phi\, \mathrm{d}x^{AB'}}{(\iota^{C'}\pi_{C'})^2},$$

where x is the intersection of Z with S_0 and y is the intersection of Z with S_1.

Example. Let h be a holomorphic function of a single (unprimed) spinor variable. Then $\phi(x) = h(x^{AA'}\iota_{A'})$ is a solution of the wave equation. In this case, the integral term in the definition of F vanishes, leaving

$$F(Z) = \frac{\phi(x)}{o^{A'}\pi_{A'}\iota^{B'}\pi_{B'}},$$

where x is the intersection point of Z with S_0. If we take S_0 to be the α-plane through the origin tangent to $o^{A'}$, then $x^{AA'} = -\mathrm{i}\omega^A o^{A'}/o^{B'}\pi_{B'}$, where $Z = (\omega^A, \pi_{A'})$. A simple application of Cauchy's theorem verifies that

$$F(\omega^A, \pi_{A'}) = \frac{h(\mathrm{i}\omega^A/o^{B'}\pi_{B'})}{o^{C'}\pi_{C'}\iota^{D'}\pi_{D'}}$$

is indeed a twistor function for ϕ.

Nick Woodhouse

References

1. M. G. Eastwood, R. Penrose, and R. O. Wells Jr: Cohomology and massless fields, *Commun. Math. Phys.* **78**, 305-51 (1981). R. S. Ward and R. O. Wells Jr: *Twistor geometry and field theory*, Cambridge University Press, Cambridge, 1990.

2. R. Penrose: Twistor theory, its aims and achievements, in *Quantum gravity, an Oxford symposium*, eds C. J. Isham, R. Penrose, and D. W. Sciama, Oxford University Press, Oxford, 1975.

3. R. S. Ward: Sheaf cohomology and an inverse twistor function, in *Advances in twistor theory*, eds L. P. Hughston and R. S. Ward, Pitman, San Francisco, 1979.