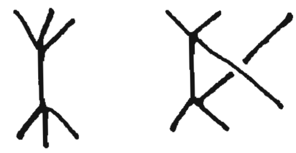
Update on Higher Order Feynman Diagrams

I can now give a much improved analysis of the second-order ϕ^4 integral discussed in **TN** 31, based on new formulas for integrating the Feynman propagator. This eliminates the guessing in that TN, but unfortunately shows my guess was wrong. I have to abandon the claim that the ultra violet divergent loop diagram

can be derived immediately from the tree diagrams



However, the general line of analysis is still valid, and the new formulas considerably extend the scope for higher-order calculations and take us nearer a transcription of general Feynman diagrams.

The basic idea is that for twistor diagram translation we need to represent Δ_F as sandwiched between test fields, i.e. to consider

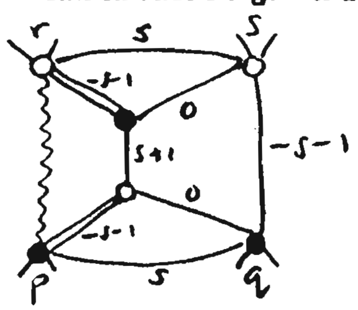
$$\int d^4x\, d^4y \;\; \phi_1(x)\, \Delta_F(x-y;\, m)\, \phi_2(y) \qquad (1)$$

where ϕ_1, ϕ_2 are unconstrained fields. It's very helpful to break this down into the cases where the test fields ϕ_i are *timelike* (i.e. positive or negative frequency) or *spacelike*. Then in the timelike case it's sufficient to consider

$$\int d^4x\, d^4y \; \frac{1}{(x-p)^2(x-q)^2} \; \Delta_F(x-y;\, m) \; \frac{1}{(y-r)^2(y-s)^2} \qquad (2)$$

where $\hat{p}, \hat{q}$, are in the past tube; $\hat{r}, \hat{s}$ in the future tube. This integral was evaluated by fairly elementary methods in my thesis long ago (1975) but I wasn't able then to see a twistor-diagram-like integral to represent it. More recent work shows that one can in fact be given as

$$\int ds \; \frac{(m^2)^s}{\sin \pi s}$$



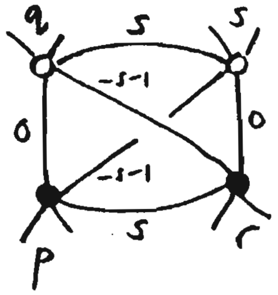
(3)

where the s-integral is a Barnes integral like this:

(The proof of this and other formulas are too involved for this brief note.)

Now consider the spacelike case, for which it's sufficient again to consider the expression (2), but now with $p^\alpha_\cdot$ $r^\alpha_\cdot$ in the past tube and $q^\alpha_\cdot$ $s^\alpha_\cdot$ in the future tube. In this case I *was* able in 1975 to give the twistor-diagram-like integral

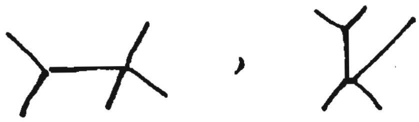
$$\int ds \frac{(m^2)^s}{\sin \pi s} \qquad (4)$$



but wasn't able to prove it correct. This has now been done. (It's a pity I didn't push my 1975 line of thought to this conclusion a long time ago.)

As a corollary, twistor diagrams for all first-order massless scattering amplitudes can be deduced via appropriate spin-raising and the limit $m \to 0$.

The formulas can also be applied to evaluate the Feynman diagrams



etc. as discussed by me in **TN** 29 and TMP, by me and L.J.O'D. in **TN** 30, and used to verify an exact correspondence with twistor diagrams.

However, the payoff I want to describe here is how they can be applied to the second-order ϕ^4 diagram

i.e. to the calculation of

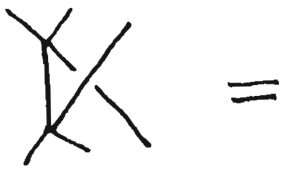
$$\int d^4x\, d^4y \frac{1}{[(x-p)^2]^2(x-s)^2} \Delta_F(x-y;0) \frac{1}{[(y-q)^2]^2(y-r)^2}$$

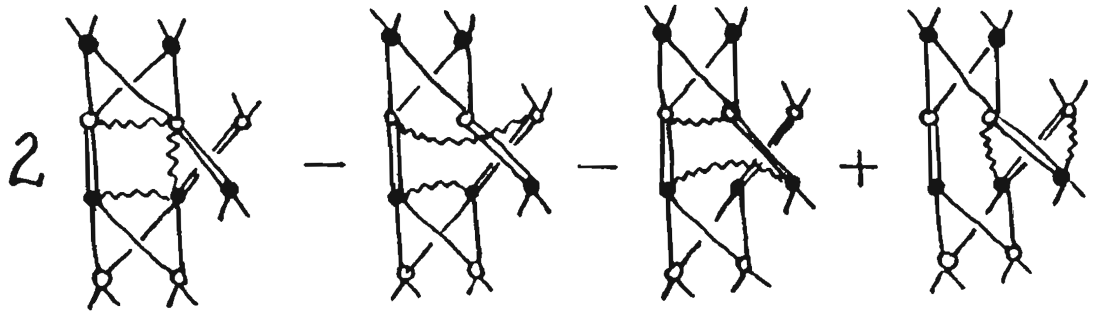
where $p^\alpha_\cdot$ r^α are in the past tube; q^α, s^α in the future tube.

This is straightforward once we observe that

$$\frac{1}{[(x-p)^2]^2(x-s)^2} = \frac{1}{[(x-p)^2]^2(p-s)^2} + \frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p}\frac{\log((p-s)^2)}{(x-s)^2(x-p)^2}$$

where the first term is manifestly timelike and the second term manifestly spacelike. Considering these two parts separately, the formulas (3) and (4) induce expressions for the two parts respectively. These expressions can in turn be represented as inhomogeneous twistor diagrams, all agreeing with the "skeleton" picture. More precisely, this must be done for a Feynman propagator with mass m, since we find that each part separately diverges as m $\to$ 0, although the sum of the two parts is convergent in this limit. The final result is that





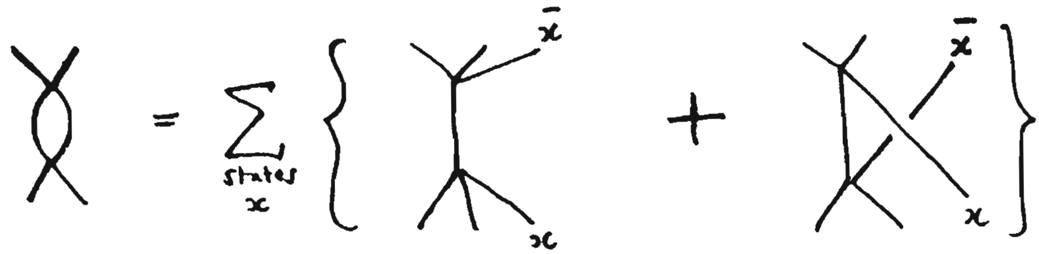
(5)

where although a mass m has to appear in the inhomogeneous boundaries at infinity, the sum of the diagrams is actually independent of m.

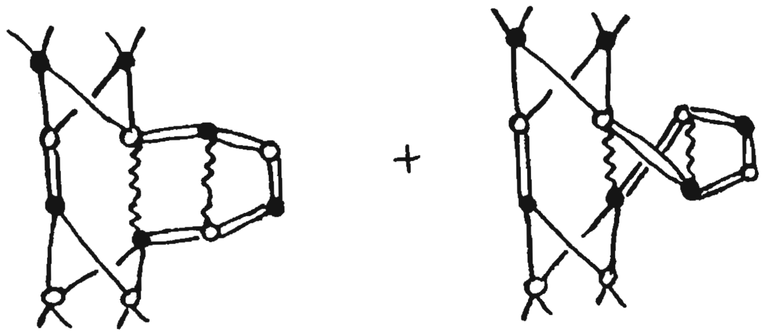
The techniques employed here could certainly be applied to other Feynman integrals. As an example, I have also computed an expression for the remaining channel

for this diagram. It appears that this too can be represented by a twistor diagram with the expected "skeleton". However, in view of the fact that (3) and (4) actually capture all the content of the Feynman propagator, it should be possible to extend these techniques much further, at least to tree diagrams.

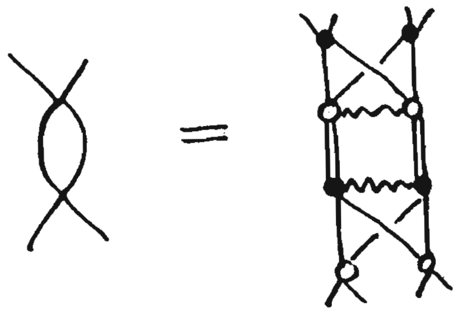
Finally there is the question of deriving the loop diagram from this result. When we now sum over external states to make the loop, as described in **TN** 31, the first term in (5) makes a contribution which is exactly cancelled by the second and third terms. Thus



is formally



In my previous note I thought that these terms would cancel each other but this now seems to be wrong. Instead, new analysis suggests that to make sense of the divergent summation, further inhomogeneous boundaries at infinity must be added, and that then we do obtain as required,



Considerable work has been done by L.J. O'D. on this calculation, developing ideas suggested in **TN** 32 about the essential role of inhomogeneity in twistor diagrams.

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