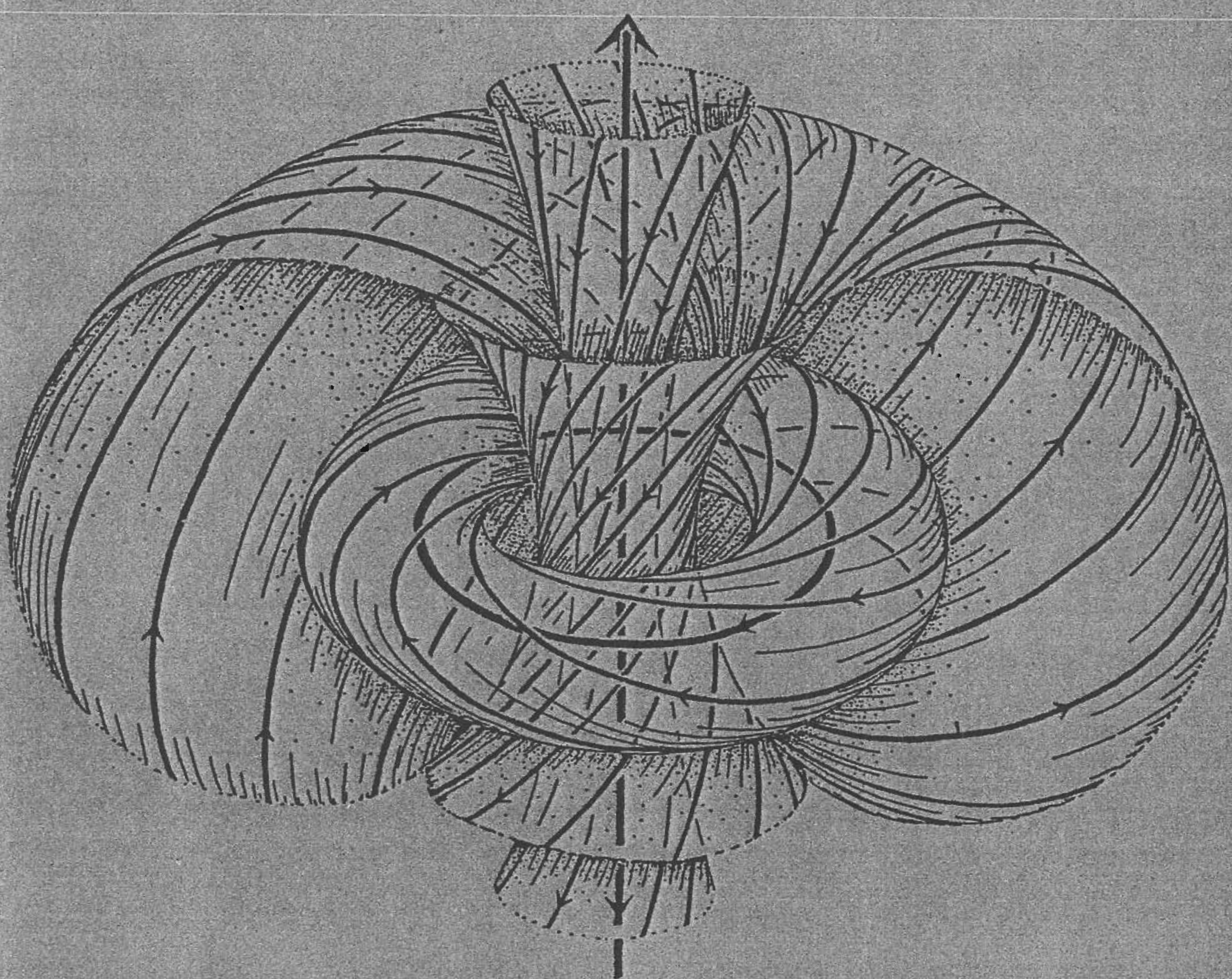
Twistor Newsletter (no 33: 5, November 1991)



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Twistors as Spin 3/2 Charges Continued: SL(3,ℂ) Bundles

In TN 32 (and in a new Festschrift volume for P.G. Bergmann), I made the suggestion that a concept of twistor, appropriate for Ricci-flat space-times generally, would be as a (conserved) charge for massless helicity 3/2 fields. This suggestion is based on the following two observations:

① $R_{ab}=0$ provides the consistency condition (in an appropriate sense) for the existence and propagation of such field, in curved space-time $\mathcal{M}$ (Buchdahl, Deser & Zumino, Julia, Chinea (-Tod));

② In Minkowski space-time $\mathbb{M}$, the space of charges for such fields is naturally identified with the twistor space $\mathbb{T}$ of $\mathbb{M}$.

Thus, it would seem that if the appropriate concept of "charge" for massless helicity 3/2 fields in a Ricci-flat $\mathcal{M}$ could be found, then this should provide the long-sought concept of twistor appropriate to the vacuum Einstein equations. By analogy with the "nonlinear graviton construction", the long-term programme is thus as follows:

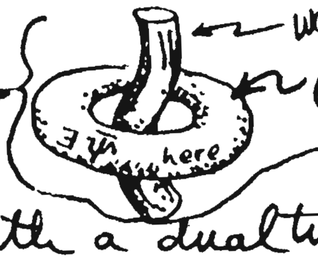
ⓐ Find this concept of twistor for Ricci-flat $\mathcal{M}$;

ⓑ Find out how to characterize the geometry of this resulting twistor space $\mathcal{T}$ — the hope being that the general such $\mathcal{T}$ can be constructed with free functions;

ⓒ Find how to reconstruct $\mathcal{M}$ from $\mathcal{T}$.

The concept of "charge" in ② above is closely analogous to the concept of energy-momentum/angular momentum that arises for linear gravity — the sources for a spin 2 massless fields in $\mathbb{M}$ (conserved). If we think of this linear gravity spin 2 field as being described by a (Weyl tensor-like) object

$$K_{abcd} = \psi_{ABCD}\,\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{A}\varepsilon_{B}\,\bar{\psi}_{A'B'C'D'}\,,$$

where $\bar{\psi}_{A'B'C'D'} = \bar{\psi}_{(A'B'C'D')}$ satisfies the massless helicity 2 field equation $\nabla^{AA'}\bar{\psi}_{A'B'C'D'}$ in some region $\mathcal{R}$ of $\mathbb{M}$ surrounding a world-tube → 

, then we can obtain the "charge" values for the tube by spin-lowering with a dual twistor twice ($2 \xrightarrow{\text{dual twistor}} \frac{3}{2} \xrightarrow{\text{dual twistor}} 1$) to obtain a self-dual Maxwell field, whose charges (electric + i× magnetic) can then be obtained by a Gauss integral, as is done in the quasi-local mass construction. Since two dual twistors are involved, the charges for linear gravity come out as the components of a symmetric valence 2 twistor $A^{\alpha\beta}$. The similar construction for a massless helicity 3/2 field $\psi_{A'B'C'}$ ($=\psi_{(A'B'C')}$; $\nabla^{AA'}\psi_{A'B'C'}=0$) involves only one dual twistor ($\frac{3}{2} \xrightarrow{\text{dual tw.}} 1$), so we get the charges as components of a valence 1 twistor, say Z^α.

For a general unrestricted $A^{\alpha\beta}$ ($=A^{(\alpha\beta)}$) there would be three types of component, with dependence upon position x^a:

quadratic in x^a — angular momentum (6 real cpts.) (primary part of $A^{\alpha\beta}$)

linear in x^a — { 4-momentum (4 real cpts.)
"magnetic" (NUT) 4-mom. (4 real cpts.) }

independent of x^a — "anti-angular momentum" (6 real cpts.) (projection part of $A^{\alpha\beta}$)

Total: 20 real cpts (= 10 complex cpts.)

But when there is an appropriate potential h_{ab} for $\psi_{A'B'C'D'}$ in R (with $\psi_{A'B'C'D'} = \frac{1}{2}\nabla^A_{(A'}\nabla^B_{B'}h_{C'D')AB}$; cf. S.&S.-T. vol.1, p.364), or, equivalently, when the sources in the world-tube come from a (linearized) energy-momentum tensor (cf. S.&S.-T. vol.2, §6.2), we have $A^{\alpha\beta}I_{\beta\gamma} = \bar{A}_{\gamma\beta}I^{\beta\alpha}$; then the "anti-angular momentum" and "magnetic" (NUT) components vanish and the order of dependence on position for the remaining components is reduced by one: linear in x^a for angular momentum; independent of x^a for 4-momentum.

All this is rather analogous to what occurs for spin $\frac{3}{2}$.
Here we have

linear in x^a — ω-part of Z^α (primary part) 4 real cpts.
independent of x^a — π-part of Z^α (projection part) 4 real cpts

Total: (8 real cpts. =) 4 complex cpts.

Somewhat analogous to the condition $A^{\alpha\beta}I_{\beta\gamma} = \bar{A}_{\gamma\beta}I^{\beta\alpha}$, arising from the existence of h_{ab} globally in $\mathcal{R}$, is the condition $Z^\alpha I_{\alpha\beta} = 0$ (i.e. $\pi_{A'} = 0 \therefore \omega^A = \text{const.}$) arising from the existence of a first potential $\gamma^A_{B'C'}$ for $\psi_{A'B'C'}$ globally throughout $\mathcal{R}$. Here we can adopt either the Dirac form [D] of the potential, for which the symmetry

[D] $\quad \gamma^A_{B'C'} = \gamma^A_{C'B'}$

holds, or else the Rarita-Schwinger form [R-S], for which this symmetry is not imposed. The equations to be satisfied are

[D] $\quad \nabla^{BB'}\gamma^A_{B'C'} = 0$

or

[R-S] $\begin{cases} \nabla^{B'(B}\gamma^{A)}_{B'C'} = 0 & \cdots\cdots ① \\ \varepsilon^{B'C'}\nabla_{A(A'}\gamma^A_{B')C'} = 0 & \cdots\cdots ② \end{cases}$

with gauge freedom $\quad \gamma^A_{B'C'} \mapsto \gamma^A_{B'C'} + \nabla^A_{B'}\nu_{C'}$,
where [D] $\quad \nabla^{AA'}\nu_{A'} = 0$

(but $\nu_{A'}$ is free in the [R-S] case). These equations also work in Ricci-flat $\mathcal{M}$.

We can also define the "field" $\psi_{A'B'C'}$ $(= \psi_{(A'B'C')})$ by

[D] $\quad \psi_{A'B'C'} = \nabla_{AA'}\gamma^A_{B'C'}$

or

[R-S] $\quad \psi_{A'B'C'} = \nabla_{A(A'}\gamma^A_{B')C'}$.

In $\mathbb{M}$, $\psi_{A'B'C'}$ is also gauge invariant and satisfies the field equation $\nabla^{AA'}\psi_{A'B'C'}$. However, for this to be true in Ricci-flat $\mathcal{M}$, we also need the condition that the Weyl tensor is anti-self-dual (ASD; i.e. $\tilde{\Psi}_{A'B'C'D'} = 0$). In that case, we can define the "π-space" for twistors by

$$\mathfrak{S}_{A'} = \{\text{glob. } \psi\text{s}\}/\{\text{glob. } \gamma\text{s}\}$$

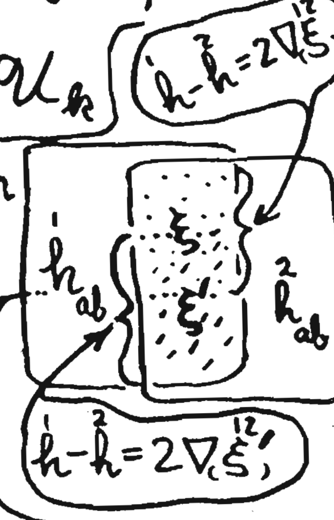
where "$\{\text{glob.}\ \hat{\psi}_s\}$" means the space of fields $\psi_{A'B'C'}$ that are global throughout $\mathcal{R}$, and correspondingly for "$\{\text{glob.}\ \gamma_s\}$". This does not work when the Weyl tensor is not ASD, however, even though there is apparently a natural-looking suggestion for a meaning for "$\{\text{glob.}\ \hat{\psi}_s\}$" in terms of a patchwork of gauge-equivalent γ_s (see R.P. in TN 32). Taking an open covering $\{\mathcal{U}_i\}$ for $\mathcal{R}$, we can try to define $\overset{i}{\gamma}$ on $\mathcal{U}_i$ where $\overset{i}{\gamma}-\overset{j}{\gamma}$ is pure gauge on each $\mathcal{U}_i \cap \mathcal{U}_j$. Unfortunately we find, in general, that any such "global ψ" is equivalent to a global γ.

It is instructive to compare this with the spin 2 case where we can consider choosing $\overset{i}{h}_{ab}$ on $\mathcal{U}_i$, to represent a linearized perturbation of the metric, with

$$\overset{i}{h}_{ab}-\overset{j}{h}_{ab}=2\nabla_{(a}\xi_{b)} \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j$$

(gauge equivalence). We find that for this to represent a genuine metric perturbation on the whole of $\mathcal{R}$, we must have, for every triple overlap

$$\overset{ij}{\xi}+\overset{jk}{\xi}+\overset{ki}{\xi}=0 \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$$



(and also, in accordance with the phenomenon for γ_s, as described in TN 32, does not count as "gauge equivalence" between $\overset{1}{h}$ and $\overset{2}{h}$ unless $\overset{12}{\xi}=\overset{12}{\xi}'$ in the central region of the overlap)).

This condition on triple overlaps serves to exclude "magnetic mass" (i.e. a NUT-type source). It is analogous to ruling out "$\pi_{A'}$-charge" in the spin 3/2 case, which is just what we do <u>not</u> want to rule out. This suggests that we do something in the spin 3/2 case that is analogous to Misner's solution to the problem of eliminating the NUT ("Dirac-type") singular wires — the triple regions on which $\overset{12}{\xi}+\overset{23}{\xi}+\overset{31}{\xi}=0$ fails. Misner noted that NUT space could be made non-singular ("wire-free") by means of identifications in the time direction that introduced closed timelike curves in the space-time. This is only possible because the NUT space-time has a timelike Killing vector, and the symmetry in that direction allows identifications to be made.

If we are to try something similar in the general spin $\frac{3}{2}$ case, we would want to erect some kind of bundle over $\mathcal{M}$ which will admit the necessary extra symmetries in the fibre direction. We also need γ to have some kind of non-linear interpretation (since we do not expect a linear π-space in the general case). The best bet appears to be to interpret γ as providing a <u>bundle connection</u>.

Take the fibre coordinates to be given by a spinor $\eta_{A'}$ and a scalar ζ. For a given small ϵ, we extend the ordinary (Christoffel-Levi-Civita) covariant derivative ∇ on $\mathcal{M}$ to bundle-valued quantities according to

$$\nabla_{PP'}\begin{pmatrix}\eta_{A'}\\ \zeta\end{pmatrix} = \begin{pmatrix}\nabla_{PP'}\eta_{A'}\\ \nabla_{PP'}\zeta\end{pmatrix} - \epsilon\begin{pmatrix}0 & \gamma_{PP'A'}\\ \gamma_{PP'}{}^{B'} & 0\end{pmatrix}\begin{pmatrix}\eta_{B'}\\ \zeta\end{pmatrix},$$

the gauge transformations being

$$\begin{pmatrix}\eta_{A'}\\ \zeta\end{pmatrix}\longmapsto\begin{pmatrix}\eta_{A'}\\ \zeta\end{pmatrix} + \epsilon\begin{pmatrix}0 & \nu_{A'}\\ \nu^{B'} & 0\end{pmatrix}\begin{pmatrix}\eta_{B'}\\ \zeta\end{pmatrix},$$

where terms of order ϵ^2 are being neglected. (There appears to be some relation to supersymmetry operations here, but as far as I can make out, what I am doing is not the same; also, an interesting limiting case of the above (studied by K.P.T.) occurs when one of the two off-diagonal entries in the matrices is set to zero, but this seems not to lead to the needed non-linearities.) As they stand, the gauge transformations do not close under commutation, and for a consistent theory valid to all orders in ϵ we need to generalize to $SL(3,\mathbb{C})$ matrices before the commutation closes. Writing $\textcircled{A}$ for the three-dimensional indices occurring here, so $\eta_{\textcircled{A}} = \begin{pmatrix}\eta_{A'}\\ \zeta\end{pmatrix}$, etc., we have a connection defined according to:

$$\nabla_{PP'}\eta_{\textcircled{A}} = \begin{pmatrix}\nabla_{PP'}\eta_{A'}\\ \nabla_{PP'}\zeta\end{pmatrix} - \gamma_{PP'\textcircled{A}}{}^{\textcircled{B}}\,\eta_{\textcircled{B}}$$

with gauge trans. $\quad \eta_{\textcircled{A}} \mapsto \eta_{\textcircled{A}} + \nu_{\textcircled{A}}{}^{\textcircled{B}}\,\eta_{\textcircled{B}}$.

The $\nu_{\textcircled{A}}{}^{\textcircled{B}}$ are $SL(3,\mathbb{C})$-valued fields on $\mathcal{M}$, so

$$\epsilon^{\textcircled{P}\textcircled{Q}\textcircled{R}}\,\nu_{\textcircled{P}}{}^{\textcircled{A}}\,\nu_{\textcircled{Q}}{}^{\textcircled{B}}\,\nu_{\textcircled{R}}{}^{\textcircled{C}} = \epsilon^{\textcircled{A}\textcircled{B}\textcircled{C}},$$

where $\epsilon^{\textcircled{P}\textcircled{Q}\textcircled{R}}$ (and $\epsilon_{\textcircled{P}\textcircled{Q}\textcircled{R}}$) are Levi-Civita objects, so

$$\epsilon^{\textcircled{A}\textcircled{B}\textcircled{C}}\,{}^{1}\!\eta_{\textcircled{A}}\,{}^{2}\!\eta_{\textcircled{B}}\,{}^{3}\!\eta_{\textcircled{C}} = {}^{1}\!\eta_{A'}\,{}^{2}\!\eta^{A'}\,{}^{3}\!\zeta + {}^{2}\!\eta_{A'}\,{}^{3}\!\eta^{A'}\,{}^{1}\!\zeta + {}^{3}\!\eta_{A'}\,{}^{1}\!\eta^{A'}\,{}^{2}\!\zeta .$$

The γs are likewise generalized, and we have an object like

$$\gamma_{PP'Ⓐ}{}^{Ⓑ} = \begin{pmatrix} \alpha_{PP'A'}{}^{B'} & \beta_{PP'A'} \\ \gamma_{PP'}{}^{B'} & \delta_{PP'} \end{pmatrix}.$$

The curvature is

$$K_{pqⒶ}{}^{Ⓑ} = 2\nabla_{[p}\gamma_{q]Ⓐ}{}^{Ⓑ} + 2\gamma_{[p|Ⓐ|}{}^{Ⓒ}\gamma_{q]Ⓒ}{}^{Ⓑ}.$$

It seems that we must consider this as a generalization of the [R-S] rather than the [D] form, since the commutators of [D] gauge quantities cease to satisfy $\nabla_A^{A'}\nu_{A'}=0$ or any other sensible local equation. One is led to speculate that an appropriate non-linear generalization of the [R-S] equations might well be

$$^{-}K_{PQⒶ}{}^{Ⓑ}=0, \quad ^{+}K_{P'Q'Ⓐ}{}^{Ⓑ}\epsilon^{P'ⒶⒸ}\epsilon_{Q'ⒷⒹ}=0$$

(or some such), where

$$K_{pqⒶ}{}^{Ⓑ} = {}^{-}K_{PQⒶ}{}^{Ⓑ}\varepsilon_{P'Q'} + \varepsilon_{PQ}{}^{+}K_{P'Q'Ⓐ}{}^{Ⓑ}$$

is the splitting of the curvature into anti-self-dual and self-dual pieces respectively. (These eqns. are respective analogues of ① and ②.) Here, $\epsilon^{P'ⒶⒸ}=\epsilon^{ⓅⒶⒸ}e_{Ⓟ}{}^{P'}$, $\epsilon_{QⒷⒹ}=\epsilon_{ⓆⒷⒹ}e_Q{}^{Ⓠ}$, where $e_{Ⓟ}{}^{P'}$ and $e_Q{}^{Ⓠ}$ are appropriate (degenerate) "soldering" quantities relating the bundle directions with tangent (spinor) directions in M, and there are presumably also equations on these, like perhaps $\nabla_{PP'}e_{Ⓟ}{}^{P}=0$. This is very much "work in progress", and the appropriate geometrical viewpoint has not fully come to light. (There are some possible relations between this type of bundle and other constructions of relevance to twistor theory. If these become clearer they will be reported on later.)

Note that the equation on $^{-}K_{\cdots}{}^{\bullet}$ tells us that we have a <u>self-dual</u> <u>connection</u> of a particular type. Thus, in the special case when M has a self-dual Weyl curvature, the Ward construction should lead to an interpretation of this connection in terms of M's <u>dual</u> twistor space, as if the charges could be defined for this connection, we have a different angle on the googly.

I am grateful particularly to Abhay Ashtekar, Lee Smolin, Ted Newman, George Sparling and Paul Tod for discussions.

R. Penrose

Special 60 minute Twistor Seminar

Oxford 1991

On a date near the beginning of August this summer the Twistor group at Oxford (minus Roger Penrose) organized a special seminar, one of the purposes of which was to exhibit the various areas of research undertaken by members of the group. Twelve speakers gave 5-minute talks, each writing down one key word on the board. The last speaker summarized our message by circling the appropriate letters. Some of the talks appear in the following pages and below is an approximate reproduction of the board at the end of the 60 minutes.

H(,*) Conf. ASD $^{+}A=0$ Particles Parabolic Integrable sYstems

Big Bang TWIstoR THEORY WORKSHOP AS DUAL YM

InteRpreting QM ChrOnology Violation SinGularity QLMomEntum

Twistor DiagRams!

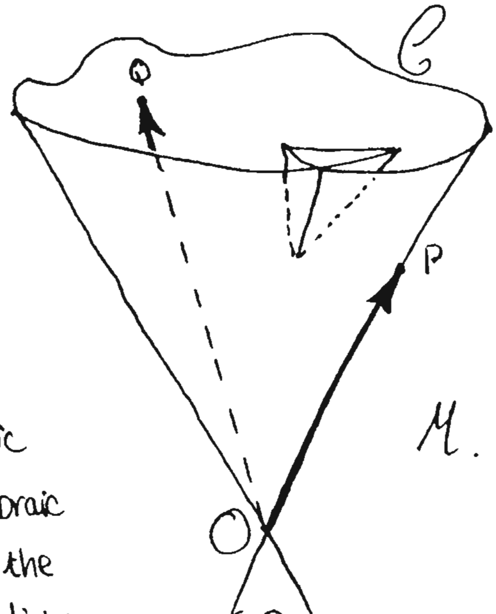
The Initial Value Problem in General Relativity by Power Series.

We work throughout in a curved vacuum space-time. At some point O, we take the light cone $\mathcal{C}$. Suppose we know the structure of $\mathcal{C}$, in the sense that we can express the component of the Weyl curvature spinor, Ψ, along any null ray. Assuming an analytic space-time we use a Taylor expansion, which when we take the component along the null ray gives, for any point P on $\mathcal{C}$,

$$\Psi_{ABCD}(P)\xi^A\xi^B\xi^C\xi^D = \Psi_{ABCD}(O)\xi^A\xi^B\xi^C\xi^D + r\,\nabla_{AA'}\Psi_{BCDE}(O)\,\xi^A\bar{\xi}^{A'}\xi^B\xi^C\xi^D\xi^E$$
$$+\frac{r^2}{2!}\nabla_{AA'}\nabla_{BB'}\Psi_{CDEF}(O)\,\xi^A\bar{\xi}^{A'}\xi^B\bar{\xi}^{B'}\xi^C\xi^D\xi^E\xi^F + \dots$$

where P is displaced by parameter value r from O along a null geodesic defined by the null vector $\xi^A\bar{\xi}^{A'}$ at O.

This form only requires the symmetric part of the derivatives of Ψ to find the component of Ψ at any point P on $\mathcal{C}$.



So we have the geometry of $\mathcal{C}$ determined by the set of all symmetric derivatives of Ψ. Now, by similarly doing a Taylor expansion for the whole space-time $\mathcal{M}$, we can express the whole Ψ at any point Q in terms of derivatives (unsymmetrised) of Ψ at O.

But we know that the symmetric derivatives of Ψ form an exact set, an algebraic basis (Penrose + Rindler, 1984), so that knowing the elements of this set (i.e. the structure of the light cone) – $\{\Psi_{ABCD}, \Psi^{E'}_{ABCDE}, \Psi^{E'F'}_{ABCDEF}, \dots\}$ where $\Psi^{E'F'\dots H'}_{ABCDEF\dots H} = \nabla^{(E'}_{(E}\nabla^{F'}_{F}\dots\nabla^{H')}_{H}\Psi_{ABCD)}$ allows us to calculate the unsymmetrized derivatives of Ψ and thus determine the structure inside the light cone. This was first suggested by Penrose in 1963 (rep. GRG 12).

We want therefore to find expressions which give the unsymmetrised derivatives of Ψ, completely in terms of those symmetrised derivatives of Ψ.

i.e. unsymmetrised n^{th} derivative of Ψ =

Ψ_n + function in lower order Ψ_n's.

where Ψ_n represents the n^{th} symmetrized derivative of Ψ.

The process of expressing the derivative of Ψ in this way will involve substituting for Ψ_n's of lower order which have already been calculated. The d'Alembertian operator on each of the Ψ_n will also be needed.

The calculation is iterative in form. Starting with the last Ψ_n calculated, a ∇ is applied and the indices symmetrised. The symmetrisation is done so that the resulting expression has all the indices still in alphabetical order. This is the canonical form which is set for spinor expressions to allow them to be simplified to their lowest form.

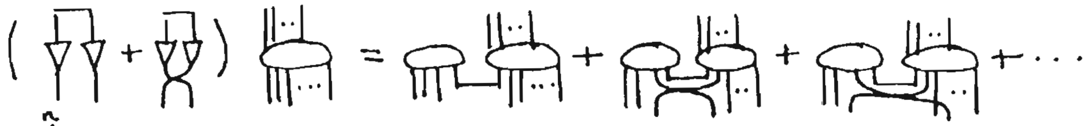
It is possible to simplify general spinor expressions by reducing them to canonical form, using the ε-identity to uncross indices.

This does produce a large number of terms. Also we need to eliminate those parts of an expression of the form $\alpha_A\alpha_B\varepsilon^{AB}$, which equal zero. This is done by symmetrising over α's indices.

$$\alpha_A\alpha_B\varepsilon^{AB} \rightarrow 1/2(\alpha_A\alpha_B\varepsilon^{AB} + \alpha_B\alpha_A\varepsilon^{BA})$$
$$\rightarrow 1/2(\alpha_A\alpha_B\varepsilon^{AB} - \alpha_A\alpha_B\varepsilon^{AB}) \rightarrow 0$$

In practice however we are symmetrising over huge numbers of indices, giving large number of terms in the expression. In order to avoid having to do too much of these simplifications, all indices are kept in canonical form as the calculation proceeds.

Continuing from the symmetrised derivative of Ψ_n, we substitute for Ψ_n. This gives an expression containing terms of at least two derivatives of Ψ_{n-1}. Terms in this expression can be simplified using the Ricci Identities, where the sum of two appropriate terms of two derivatives is replaced with a product of Ψ's, i.e.



where  $= \Psi_n$.

There will also be terms of the form  to be substituted for. This will use an expression for $\Box \Psi_{n-1}$ which will have been calculated at the same time as Ψ_{n-1}, where

$$\Box \Psi_{n-1} = \sum \text{ terms in } \Psi_{n-1} \text{ and lower order } \Psi\text{'s}.$$

The expression can thus be reduced to one containing only Ψ_n's and one term, the unsymmetrised n^{th} derivative of Ψ_{ABCD}. These methods are also applied in calculating $\Box \Psi_{n-1}$ as part of the iterative process.

Starting with Ψ_{ABCD} and knowing that $\varepsilon^{AB} \nabla_{AA'} \Psi_{BCDE} = 0$, we can easily calculate $\Psi_1 = \nabla_{AA'} \Psi_{BCDE}$ and can calculate $\Box \Psi_{ABCD}$ in terms of Ψ_{ABCD}. This is then take to calculate Ψ_2 in terms of Ψ_1, Ψ_0 plus $\nabla_{AA'} \nabla_{BB'} \Psi_{CDEF}$, and $\Box \Psi_1$ in terms of Ψ_1 and Ψ_0, and so on.

This process has been implemented in Mathematica. The expressions obtained are extremely long and Mathematica encounters certain difficulties in managing expressions of this length effectively. Explorations are being made to transfer this work to another system.

A more specialised approach to the problem would be to set the light cone to be converging, by giving specific values for the symmetric derivatives which set the structure of ℓ. This can be achieved by,

$$\Psi_{ABCD} = a\, \iota^A \iota^B \iota^C \iota^D \quad \text{and} \quad \Psi^{E'}{}_{ABCDE} = b\, o^{E'} o_E o_A o_B o_C o_D$$

and all others are zero.

This reconverging light cone structure would mean that there must be a singularity in the space-time. The structure of the space-time as the singularity is approached could be investigated using this method.

Vanessa Thomas

Conformal Singularities and the Weyl Curvature Hypothesis

Richard P.A.C. Newman*

Abstract *A conjecture of K.P. Tod relating Penrose's Weyl Curvature Hypothesis and the isotropy of the Universe has recently been proved. Background material and the method of proof are discussed.*

The large scale structure of the Universe is well-known to exhibit a high degree of spatial isotropy. Claims have been made that this may be due to quantum processes occurring at times subsequent to the Big Bang, such as inflation and dissipative particle interactions. According to Penrose (1977) however, one should seek a more fundamental explanation in terms of entropy considerations at the Big Bang itself. This view has recently been provided mathematical substantiation by a development in classical general relativity.

Although Penrose is concerned to apply a condition of zero entropy at the Big Bang, it is worthwhile to first recall a contrary view of Misner (1968) that the Big Bang was maximally disordered. One might seek to motivate this philosophy of 'chaotic cosmology' on the grounds that physical laws based on field equations are inherently inapplicable at singularities, and that the Big Bang, being singular, cannot therefore be constrained by such laws. As will be seen, this view may be naive. But whatever its motivation, the major problem for chaotic cosmology is to explain how the high degree of large-scale isotropy of the present-day Universe could have evolved from an chaotic Big Bang. Neutrino-induced viscosity (Matzner and Misner (1972)) and curvature-induced creation of particle pairs (Zel'dovich (1972)) are just two of the possible isotropization mechanisms that have been investigated. However further studies (Collins and Stewart (1971), Collins and Hawking (1973), Barrow and Matzner (1977)) indicate that that the age and expansion rate of the Universe place such severe constraints on the effects of such processes that gravitational

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instabilities would have been likely to predominate, causing the growth of any initial anisotropies. Chaotic cosmology therefore appears to be untenable:

Chaos then implies chaos now.

Consider then the alternative view that the Big Bang was ordered. One might reasonably hope that such order would be found to have its origins in the laws of physics rather than be a manifestation of initial conditions imposed at the will of a Creator. But in any case, the principal problem for 'quiescent cosmology' (Barrow (1978)) is to show that an appropriately ordered Big Bang necessarily evolves to a spatially isotropic Universe. That is:

Order then implies order now.

This problem breaks naturally into two parts. First, to determine a suitable condition of order at the Big Bang. And second, to establish the uniqueness of the evolution despite the singular nature of the initial state. The problem, as a whole, was given a mathematically clear formulation in the context of classical general relativity by Tod (1987). His fundamental proposal was that the Big Bang be assumed to be a conformal singularity, meaning that it could be conformally transformed into a smooth spacelike hypersurface. With regard to the condition of order at the Big Bang, he adopted Penrose's Weyl Curvature Hypothesis (Penrose (1977), (1981), (1986)) that the Weyl tensor tends to zero at the Big Bang. This Hypothesis is motivated by a speculation that the Weyl tensor may be related to an, as yet undefined, gravitational entropy which is initially zero, but increases as gravitational instabilities, and ultimately gravitational collapse, come into effect. This in turn is motivated firstly by calculations which show that singularities resulting from gravitational collapse have a tendency towards high anisotropy, and correspondingly large Weyl tensor, and secondly by a need (Penrose (1979)) to account for the temporal assymmetry of the Universe. But to return to the problem in hand, Tod imposed a final condition that the Universe be assumed to be filled by an isentropic perfect fluid. His conjecture then was that, subject to the stated conditions, the Universe is necessarily spatially isotropic and therefore Robertson-Walker.

The study of conformal singularities arose out of a well-known series of papers on singularities in general relativity by Lifschitz and co-workers, beginning with Lifschitz and Khalatnikov (1963) and culminating with Belinski et al. (1982). Unfortunately, in view of their use of power series approximations, not all their conclusions can be considered well-founded. Nonetheless, their work did indicate that, in modern terminology, the induced conformal 3-metric on the initial hypersurface representing a conformal singularity determines the 4-dimensional Riemann tensor there. Subsequent calculations of Goode and Wainwright (1985) (who employed the term 'isotropic singularity') enabled Tod (1987) to conclude that the Weyl Curvature Hypothesis implies that the conformal 3-metric on the initial hypersurface is of constant curvature, and therefore isotropic and likely to give rise to a Robertson-Walker Universe. Certainly any constant curvature 3-metric can arise as the initial conformal 3-metric for some Robertson-Walker model. The outstanding problem was therefore to show that the initial conformal 3-metric constitutes sufficient initial data to uniquely determine the subsequent evolution. This has now been carried out for the special case $\gamma = 4/3$ corresponding to a Universe filled by radiation or a highly relativistic fluid. (The adiabatic index γ, in general a function of the fluid density ρ, determines the pressure p of the fluid according to $p = (\gamma - 1)\rho$.)

Theorem (Newman (1991). A $\gamma = 4/3$ perfect fluid space-time which evolves from a spacelike conformal singularity subject to the Weyl Curvature Hypothesis is necessarily Robertson-Walker near the singularity. ■

This result gives an affirmative resolution of the Tod conjecture in the case $\gamma = 4/3$. It is likely that a routine extension of the techniques involved in the proof may permit more general equations of state, although asymptotic conditions on γ for large matter density ρ will undoubtedly be necessary.

One technical point should be made. Previous authors, and Goode and Wainwright (1985) in particular, permitted or even implicitly demanded the conformal factor in the description of the conformal singularity to be non-differentiable at the initial hypersurface representing the singularity. This gave rise to considerable complications. For present purposes however, the conformal factor is assumed, along with the conformal manifold and the conformal metric, to be C^∞. This has the curious, but desirable consequence that, as the singularity is approached, γ must tend to the value $4/3$ appropriate to a hot Big Bang. For mathematical simplicity the theorem assumes that γ has what is therefore its only possible constant value in this context, namely $\gamma = 4/3$.

Although the proof of the theorem is long, the underlying method is sufficiently straightforward to be outlined here. The first task is to fix the conformal factor. It can be shown (Scott (1988)) that the velocity of the fluid is necessarily irrotational in both the physical and conformal pictures, and meets the initial hypersurface orthogonally in the latter. One can therefore demand that the conformal factor be such that its level surfaces are orthogonal to the fluid velocity. The conservation equations then suggest a natural scaling in relation to the fluid density ρ . Following Goode and Wainwright (1985) one now shows that, within the conformal picture, the magnetic part of the Weyl tensor must vanish at the initial hypersurface, whilst the electic part is proportional to the trace-free part of the Ricci tensor of the conformal 3-metric thereon. The Weyl Curvature Hypothesis and the contracted Bianchi identities are now sufficient to show that this conformal 3-metric is of constant curvature.

The irrotationality of the fluid velocity suggests the use of comoving coordinates. However it is well known that, at least for the vacuum Einstein equations, hyperbolicity and the consequent well-posedness of the Cauchy problem are most easily demonstrated in harmonic coordinates. Nonetheless, comoving coordinates turn out to be the better choice. Independent variables are now selected in such a manner as to obtain from the conformally transformed Einstein equations for the fluid a first order quasi-linear symmetric hyperbolic system of evolution equations of the form:

$$A^0(u)\,\partial_t u = A^i(u)\,\partial_i u + (B(u) + t^{-1}C(u))u\,.$$

$$u = \mathring{u} \quad \text{at} \quad t = 0$$

Here u is a 63-component column vector, $A^0(u), \ldots, A^3(u)$, $B(u)$ and $C(u)$ are 63×63 matrices, analytic in u , with $A^0(u), \ldots, A^3(u)$ symmetric (hence the 'symmetry' of the system) and $A^0(u)$ positive definite (hence the 'hyperbolicity'). Also $(A^0(u))^{-1}C(u)$ has no positive integer eigenvalues. The quantity t is the conformal factor which also serves as a time coordinate. The spatial coordinates are labelled by $i = 1, 2, 3$. The function $\mathring{u}$ on the initial hypersurface is fixed by the conformal 3-metric there together with the gauge conditions. One can show that the system not only follows from, but is equivalent to the original conformally transformed Einstein equations from which it was derived. However this is not relevant to the Tod conjecture for which it would suffice to work with any of a number of smaller systems that are not known to posess this property.

In order to complete the proof, it remains to establish a uniqueness theorem for solutions to equations of the above form. Somewhat surprisingly, no suitable theorem was to be found in the literature, so a special study had to be undertaken. The property that $(A^0(u))^{-1}C(u)$ has no positive integer eigenvalues plays a fundamental role. To see this, suppose for simplicity that one wishes to establish uniqueness only amongst analytic solutions. For any such solution u, the above equation may be differentiated $n-1$ times and restricted to $t=0$ to yield

$$(\mathrm{Id} - n^{-1}(A^0(0))^{-1}C(0))\,\partial_t^n|_{t=0}u \;=\; \text{terms in } \partial_t^p|_{t=0}u \text{ and } \partial_t^p|_{t=0}\partial_i u$$

$$i = 1, 2, 3\,; \quad 0 \le p \le n-1\,; \quad n \ge 1$$

$$u = \overset{\bullet}{u} \quad \text{at} \quad t = 0$$

For each positive integer n one can thus, by induction, express $\partial_t^n|_{t=0}u$ in terms of $\partial_i^p \overset{\bullet}{u}$, $0 \le p \le n-1$, $i = 1, 2, 3$. It follows that u, being analytic, is uniquely determined on a neighbourhood of the initial hypersurface $t = 0$ in terms of $\overset{\bullet}{u}$ and its derivatives. One can in fact not only dispense with analyticity, but work at finite levels of differentiability by means of fixed point techniques. Even at the C^∞ level some subtlety is required because the coefficient t^{-1} in the basic system of equations tends to resist contraction mappings. The difficulties can be overcome however to yield the required uniqueness theorem and hence a proof of the Tod conjecture in the case $\gamma = 4/3$.

An outstanding problem is to show that the conformal 3-metric on the initial hypersurface constitutes a complete set of initial data, even without the imposition of the Weyl Curvature Hypothesis. To have obtained a symmetric hyperbolic system equivalent to the original conformally transformed Einstein equations is a significant advance. A suitable existence theorem for this system would complete the result.

The work described here concerning perfect fluid space-times in the vicinity of conformal singularities in many ways parallels work of Friedrich (1985) concerning vacuum space-times in the vicinity of conformal infinity. Symmetric hyperbolic systems form the key ingredient in both instances, although in the case of conformal infinity one does not have to contend with a t^{-1} forcing term. Perhaps a unified treatment may be possible. But in any case it is a remarkable feature of the Einstein equations, with or without matter, that the fundamental property of hyperbolicity can survive conformal transformations, even where the conformal factor passes through zero or infinity. Whilst the mathematics of this phenomenon is part way to being understood, the underlying physical significance remains mysterious.

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Twistor theory and integrability

This note consists mostly of certain speculative and conjectural comments that I gave or would have liked to give in my 5 minute contribution to the special twistor workshop to celebrate the birthday of the founder of many of these ideas.

In this note I wish to emphasize that the recently established deep links between twistor theory and integrable systems should lead to new techniques and results in twistor theory as well as unification and hopefully new results in the theory of integrable systems.

In various articles it has emerged that many/most integrable systems are symmetry reductions of the self-dual Yang-Mills equations with a small number of exceptions, most notable of which is the KP hierarchy. Furthermore, much of the theory and structure of these equations can be understood in a reasonably direct way as various features of the symmetry reductions of the Ward correspondence for the self-dual Yang-Mills equations. See Mason & Sparling and references therein.

As far as the equations are concerned, it appears that we can classify most integrable systems as reductions from self-dual Yang-Mills in 4-dimensions by choice of:

a) a gauge group,

b) a symmetry group (with a possible discrete component),

c) a normal form for the various constants of integration that arise in the reduced equation.

For example, the Drinfeld Sokolov systems can all be understood in this way as can various other large classes of integrable systems.

The standard theory of the equations consists of such constructions as the inverse scattering transform, actions of loop groups and realizations of the P.D.E.s as flows on grassmanians. These can be understood as various ansätze and normal forms for the patching data of the holomorphic vector bundles on twistor space with the appropriate symmetry properties that arise from the corresponding symmetry reductions of the Ward transform for the self-dual Yang-Mills equations.

However, these ideas from the theory of integrable systems are in many cases refinements of twistor ideas, and in others are completely new in twistor theory. There is therefore the possibility of methods from the theory of integrable systems being used to solve problems in twistor theory. The following conjectures and connections include examples where twistor theory may benefit from this interaction.

1) Inverse scattering. The inverse scattering transform provides a parametrization of the solution space of integrable P.D.E.s. The parameters can be used to build patching data for the Ward bundle on twistor space directly. For example for the attractive nonlinear Schrodinger equation we get the solution space identified with: $Map(S^1 \mapsto D) \times \amalg_{k=1}^{\infty} S^k\{C^* \times D\}$ where D is the unit disc in the complex plane C, C^* are the non-zero comlex numbers, $\amalg$ is the disjoint union and S^k is the symmetrized cartesian product. The first factor are solutions that one would expect from linearizing the equations (for which they are the Fourier transform) but the second factor are the soliton solutions which do not have a linear analogue.

One would expect this pattern to be generic for solutions of the self-dual Yang-Mills equations in indefinite signature. So one would expect for example that on the compactified 4-dimensional Minkowski space with signature (2,2) the solution space of the $SU(n)$ self-dual Yang-Mills equations is a Cartesian product of maps from RP^3 to unit determinant Hermitean $n \times n$ matrices with a soliton type sector, which would presumably be the (2,2) analogues of instantons. It is perhaps worth mentioning that the first factor can be understood as a nonlinear generalization of the Radon transform. A similar picture should hold for the symmetry reductions to equations in $2+1$ dimensions and other $1+1$ dimensional systems.

2) The inverse scattering transform in 2+1 dimensions such as for the KP hierarchy has features that distinguish it clearly from existing twistor correspondences so that there seems little real hope of incorporating it into the above framework. Nevertheless, it is a natural generalization of the framework for the KdV equations and leads to a coherent inverse scattering transform based on a non-local Riemann-Hilbert transform. One may hope, then, that the transform can be articulated geometrically so that it leads to some new category of twistor constructions.

It is perhaps worth remarking that the pseudo-differential operators that play such a prominent role in the KP equations also arose naturally in one of RP's earlier discussions of the googly problem—the patching operation was represented by a pseudo-differential operator representable by integration against a kernel just as in the KP inverse scattering problem.

Another point is that the inverse scattering transform does work for many other field equations in higher dimensions but is no longer implementable by linear procedures and hence does not lead to practical solution generation methods. It may nevertheless lead to a workable framework for understanding general relativity using spin 3/2 fields and RP's elemental states based on asymptotic twistors (see the previous TN).

3) It is possible to use the Ward correspondence to understand the connections between the KdV type equations and the 2-dimensional quantum field theory of free Fermions, developed by the Japanese school and described in Segal & Wilson and Witten. Solutions (at least those that are reflectionless) of the KdV equations are given by amplitudes associated to flows acting on certain special vectors in the free Fermion Fock space. The link is that the free Fermions are the holomorphic sections of the Ward bundle on twistor space restricted to a complex projective line, and the quantum field theoretic amplitude in question is the 2-point function that gives rise to the Greens function for the $\bar{\partial}$-operator. Finding the Greens function is equivalent to trivializing the vector bundle on the line which is the key step in obtaining the self-dual Yang-Mills field in terms of the bundle.

One may ask the question then of whether its possible to realize other more complicated twistor constructions such as the nonlinear graviton construction as a more complicated, perhaps interacting 2-dimensional quantum field theory. In particular this might explain the remarkable link discovered by Ooguri & Vafa between $N = 2$ string theory and the self-dual Einstein equations.

4) There is much scope for using ideas from the quantum inverse scattering transform to understand how to use twistor methods in the context of integrable quantum field theory. In particular the Russian school's introduction of the R-matrix to describe the Poisson bracket structure should pass over directly to give the Poisson bracket relations for the twistor patching data. Other workers have managed to show that the inverse scattering transform survives quantization so that one can hope to quantize on twistor space and then transform the results to obtain a quantum field theory on space-time. The existing theory is still in need of further insights that twistor theory may be able to provide.

5) Witten has attempted a unification of the theory of integrable statistical mechanical models using Chern-Simons quantum field theory. This produces R-matrices, and is sufficient for understanding knot polynomials. Unfortunately it does not provide the dependence of the R-matrices on the spectral parameter that is so crucial to integrability. So it is not possible to regard this as a satisfactory understanding of integrable statistical mechanics. One may conjecture that by studying a quantum field theory of self-dual Yang-Mills reduced to 3-dimensions this gap would be remedied.

It is perhaps also worth drawing attention to the Atiyah-Murray conjecture also in this context (see their article in the last TN).

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The 3-Wave Interaction from the Self-dual Yang Mills Equations

There is a family of completely integrable systems called 'the n-wave interaction' (see eg Ablowitz and Segur 1981). According to current twistor dogma, these equations should be reductions of the self-dual Yang-Mills equations. While trying to do something different, I found a way of getting them by this route. I also found, though somewhat later, that Chakravarty and Ablowitz (1990) had a slightly different route with similar end-points.

The starting point is the self-dual Yang-Mills equations with 2 null symmetries. These are equivalent to the commutation relation

$$[D_1, D_2] = 0 \qquad (1)$$

where

$$D_1 = \partial_1 - A_1 + \zeta B_1$$
$$D_2 = \partial_2 - A_2 + \zeta B_2, \qquad (2)$$

the A_i and B_i are n×n complex matrices, functions of x^1 and x^2 only, and ζ is a complex constant.

Substituting (2) into (1) and equating separate powers of ζ to zero gives 3 equations. The $O(\zeta^2)$ term is just

$$[B_1, B_2] = 0 \qquad (3)$$

Mason and Singer (1991; see also Mason 1991) solve this by taking the B_i to be nilpotent and arrive at the n-th generalised KdV equation. The opposite extreme, which I shall take, is to suppose that the B_i are diagonalisable by the Yang-Mills gauge freedom, which is

$$B_i \to G^{-1}B_iG \; ; \; A_i \to G^{-1}(A_iG - \partial_iG). \qquad (4)$$

where G is an n×n complex matrix depending on x^1 and x^2.

Now the $O(\zeta)$ term in (1) is

$$\partial_1B_2 - \partial_2B_1 + A_2B_1 + B_2A_1 - A_1B_2 - B_1A_2 = 0 \qquad (5)$$

The diagonal entries in (5) imply that there is a 'potential' for the B_i:

$$B_i = \partial_iC \qquad (6)$$

while the off-diagonal entries imply that the off-diagonal entries of the A_i are proportional in a way that I shall write out explicitly below. Before that, we consider the O(1) term in (1) which is

$$\partial_2A_1 - \partial_1A_2 + A_1A_2 - A_2A_1 = 0 \qquad (7)$$

The diagonal entries in (7) imply that the diagonal entries of the A_i have potentials in a way analogous to (6). A gauge transformation (4) with diagonal G preserves the diagonality of the B_i and can be chosen to remove the diagonal entries of the A_i.

To summarise the situation at this point in the argument:

(i) the B_i are diagonal and derived from a potential C as in (6);
(ii) the A_i are purely off-diagonal and can be expressed in terms of C and each other using (5);
(iii) finally (7) imposes some differential equations.

At what is essentially this point, Chakravarty and Ablowitz (1990) take the matrices B_i to be constant and arrive at the n-wave interaction. This is a specialisation in that the B's can't in general be made constant by a gauge transformation (4), but it leads to the same equations eventually as we shall see.

Now it is necessary to resort to taking components so for simplicity I will restrict to 3×3 matrices. Set

$$B_1 = \mathrm{diag}(\alpha,\beta,\gamma) = \partial_1 C \; ; \qquad B_2 = \mathrm{diag}(\lambda,\mu,\nu) = \partial_2 C \qquad (8)$$

and
$$\alpha - \beta = \partial_1 P \qquad \beta - \gamma = \partial_1 Q \qquad \gamma - \alpha = \partial_1 R \qquad (9)$$
$$\lambda - \mu = \partial_2 P \qquad \mu - \nu = \partial_2 Q \qquad \nu - \lambda = \partial_2 R$$

so that
$$P + Q + R = 0. \qquad (10)$$

We will eventually switch to using two of P,Q,R as independent variables.

With the choices (8) for the B_i, we can solve (5) for the A_i in terms of another off-diagonal matrix E. Set

$$A_1 = (a_{ij}) \quad A_2 = (b_{ij}) \quad E = (e_{ij})$$

then (5) implies

$$a_{12} = (\alpha-\beta)e_{12} \; ; \; b_{12} = (\lambda-\mu)e_{12} \qquad (11)$$

and the 5 equations obtained from this by the obvious permutations.

Finally, we substitute (11) into (7) to obtain differential equations on E. These differential equations can all be written with the aid of the Poisson bracket in (x^1,x^2). The typical one, from which the other 5 follow by permutations, is

$$\{E_{12},R\} = E_{13}E_{32}\{P,Q\} \qquad (12)$$

We can break the symmetry between P,Q,R by adopting P and Q as new independent coordinates. Write 'dot' and 'prime' for differentiation w.r.t P and Q respectively and set

$$E = \begin{pmatrix} 0 & H & V \\ W & 0 & F \\ G & U & 0 \end{pmatrix}$$

then (12) becomes the system

$$
\begin{aligned}
F' &= -VW & \qquad U' &= GH \\
\dot{G} &= WU & \qquad \dot{V} &= -FH \\
\dot{H} - H' &= -UV & \qquad \dot{W} - W' &= FG
\end{aligned}
\tag{13}
$$

which is equivalent to the 3-wave interaction.

The further reduction 'dot = minus prime' leads, after some manipulating of constants, to the integrable Hamiltonian

$$h = p_1 p_2 q_3 + q_1 q_2 p_3$$

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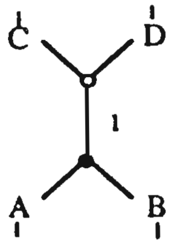
Paul Tod

Twistor diagrams in higher dimensions

The single box in higher dimensions

The observation that integration of homogeneous twistor-differential forms (twistor diagrams) over chosen $\mathbb{Z}$-cycles [C], such as for example

$$\int_{\mathbb{C}^4\times\mathbb{C}^{4*}\supset C_1,\,C_2}$$ 

$$= \pm \begin{cases} \left(\dfrac{\begin{smallmatrix}A & B\\ \mid & \mid\end{smallmatrix}}{\begin{smallmatrix}\mid & \mid\\ C & D\end{smallmatrix}}\right)^{-1} \lg \begin{smallmatrix}A & B\\ \mid & \mid\\ C & D\end{smallmatrix} \Big/ \begin{smallmatrix}A & B\\ \mid & \mid\\ D & C\end{smallmatrix} & (C_1) \\ \left(\dfrac{\begin{smallmatrix}A & B\\ \mid & \mid\end{smallmatrix}}{\begin{smallmatrix}\mid & \mid\\ C & D\end{smallmatrix}}\right)^{-1} & (C_2) \end{cases} \qquad (1)$$

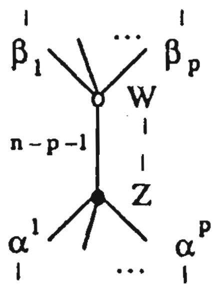
result in expressions which make sense in any dimension (e.g.

$$\underset{\mid}{A},\ \underset{\mid}{B},\ \overset{\mid}{C},\ \overset{\mid}{D} \in \mathbb{C}^{n\,(*)}\ ,\quad n \geq 2\ ) \qquad (2)$$

raises the natural question:

Are there higher (or in some cases also lower) dimensional forms and cycles (contours) which reproduce those same results?

In pursuit of this question – which will also have interesting implications for the inversion of differential operators corresponding to massless propagators – we start with the following procedure of "inflating" a particular contour $C_{n,p}$ along the internal line of a twistor diagram on $\mathbb{C}^n{}_Z \times \mathbb{C}^{n*}{}_W$:



$$\longleftrightarrow \quad \frac{(n-p-1)!\, D^n Z\, D^n W}{\begin{smallmatrix}\alpha_1 & & \alpha_p\\ \mid & \cdots & \mid\\ Z & & Z\end{smallmatrix}\left(\begin{smallmatrix}W\\ \mid\\ Z\end{smallmatrix}\right)^{n-p}\begin{smallmatrix}W & & W\\ \mid & \cdots & \mid\\ \beta_1 & & \beta_p\end{smallmatrix}} =: \ \omega_n \qquad (3)$$

$$D^n Z = dZ \ldots dZ / n! \quad , \quad \begin{smallmatrix}\alpha^1 \ldots \alpha^p\\ \vdash\!\!\dashv\\ \beta_1 \ldots \beta_p\end{smallmatrix} \neq 0 \quad , \quad p < n \ .$$

We extend the exterior parameters to bases

$$\{\underset{\mid}{\alpha^1},\ldots,\underset{\mid}{\alpha^n}\},\ \{\overset{\mid}{\beta_1},\ldots,\overset{\mid}{\beta_n}\} \subset (\mathbb{C}^n_Z)^*,\ \mathbb{C}^n_W \quad \text{with} \quad \begin{smallmatrix}\alpha^i\\ \mid\\ \beta_j\end{smallmatrix} = \delta^i_j \quad \text{for } \max\{i,j\} > p\ ;$$

with their duals

$$\{\overset{\mid}{\alpha_1},\ldots,\overset{\mid}{\alpha_n}(=\overset{\mid}{\beta_n})\} \subset \mathbb{C}^n_Z\ (\equiv \mathbb{C}^n_W),\ \{\underset{\mid}{\beta^1},\ldots,\underset{\mid}{\beta^n}(=\underset{\mid}{\alpha^n})\} \subset \mathbb{C}^{n*}_W\ (\equiv (\mathbb{C}^n_Z)^*)$$

so that

$$Z = \sum_{i=1}^{n} \frac{\alpha^i}{Z}\, \alpha_i = \Sigma\, z^i \alpha_i \;, \quad W = \Sigma \frac{W}{\beta_i}\, \beta^i = \Sigma\, w_i \beta^i \quad . \tag{4}$$

Integration of ω_n over the standard cycle $[C_{n,p}] = [(S^1)^{2p+1} \times S^{2n-2p-1}]$ with

$$(S^1)^{2p+1} \times \underbrace{S^{2n-2p-1}}_{S^1 \times_{Hopf} \mathbb{C}P^{n-p-1}} = \left\{ \begin{matrix} ((z^1, \ldots, z^n), \\ (w_1, \ldots, w_n)) \end{matrix} \;\middle|\; \begin{matrix} z^j = \varepsilon e^{i\phi_j},\ w_j = \varepsilon e^{i\psi_j},\ j \le p\,; \\ w_j = \bar{z}_j e^{i\phi},\ \Sigma z^j \bar{z}_j = 1,\ j > p\,; \\ \phi,\ \phi_j,\ \psi_j \in [0, 2\pi],\ \varepsilon \text{ const} \ll 1 \end{matrix} \right\} \tag{5}$$

gives

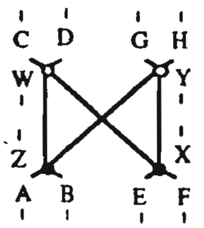
$$\int_{C_{n,p}} \omega_n = (2\pi i)^{n+p+1} \left(\frac{\alpha^1 \;\ldots\; \alpha^p}{\beta_1 \;\ldots\; \beta_p} \right)^{-1} \quad . \tag{6}$$

The contour $C_{n,p}$ and the differential form ω_n can obviously be constructed for any $n > p$ (and indeed also for $n = p$). In the following commuting embeddings we regard $C_{m,p}$ for $m \ge n$ as the "inflated" version of the contour $C_{n,p}$:

$$\begin{matrix} C_{n,p} & \hookrightarrow & C_{m,p} \\ \downarrow & & \downarrow \\ \mathbb{C}^n \times \mathbb{C}^{n*} & \hookrightarrow & \mathbb{C}^m \times \mathbb{C}^{m*} \end{matrix} \quad . \tag{7}$$

To get contours B_n for higher dimensional box diagrams we apply this procedure twice, first to the base and then to the fibres over this "inflated" base of a standard contour B_0 for the box diagram in

$\mathbb{C}^4{}_Z \times \mathbb{C}^{4*}{}_W \times \mathbb{C}^4{}_X \times \mathbb{C}^{4*}{}_Y$:



$$\longleftrightarrow \quad \omega_{0,0} = \frac{D^4Z\, D^4W\, D^4X\, D^4Y}{\begin{matrix} \text{YAB WWW WEF YYY} \\ \text{ZZZZ C D XXXX GH} \end{matrix}} \;, \quad \underbrace{\frac{AB}{CD}\,\frac{EF}{GH}}_{=:\,a}\;\underbrace{\frac{AB}{GH}\,\frac{EF}{CD}}_{=:\,c} \ne 0 \tag{8}$$

If the ingoing and outgoing states are not too far apart

$$\frac{AB}{} \approx \frac{EF}{} \;, \quad \frac{}{GH} \approx \frac{}{CD} \tag{9}$$

B_0 can be described as a fibre bundle

$$\begin{array}{ccccccc} (S^1)^8 & \sim & C_{4,3} & \sim & F_0 & \longrightarrow & B_0 \\ & & & & & & \downarrow \\ & & & & & & \pi_0 B_0 \sim C_{4,2} \sim (S^1)^5 \times S^3 \end{array} \qquad (10)$$

where π_0 is the projection onto the factor $\mathbb{C}^4{}_X \times \mathbb{C}^{4*}{}_Y$. The base space is constructed to be a $C_{4,2}$ as in (5) using bases as in (4)

$$\{ \underset{|}{\alpha}^1,\ldots,\underset{|}{\alpha}^4 \},\ \{ \overset{|}{\beta}_1,\ldots,\overset{|}{\beta}_4 \} = \{ \underset{|}{E},\underset{|}{F},\underset{|}{\alpha}^3,\underset{|}{\alpha}^4 \},\ \{ \overset{|}{G},\overset{|}{H},\overset{|}{\beta}_3,\overset{|}{\beta}_4 \}\ ;\ \begin{matrix}\alpha^i \\ | \\ \beta_j\end{matrix} = \delta^i_j \quad \text{for } \max\{i,j\} > 2\ ; \qquad (11)$$

with the additional diagonalising property

$$\frac{\overset{|\,|\,|}{AB\alpha^i}}{\underset{|\,|\,|}{CD\beta_j}} = \frac{\overset{|\,|}{AB}}{\underset{|\,|}{CD}}\,\mu(i)\,\delta^i_j \quad \text{for } i,j > 2. \qquad (12)$$

(9) then implies that $\mu(3), \mu(4) \approx 1$ and hence

$$\left| \frac{\overset{|\,|\,|}{ABX}}{\underset{|\,|\,|}{CDY}} \Big/ \frac{\overset{|\,|}{AB}}{\underset{|\,|}{CD}} \right| \approx 1 + O(\varepsilon) \neq 0 \qquad \text{for all } (\overset{|}{X},\underset{|}{Y}) \in \pi_0 B_0\,. \qquad (13)$$

This allows us to fix the fibres $F_{0\,(\overset{|}{X},\underset{|}{Y})}$ as standard $C_{4,3}$'s ($\sim (S^1)^8$) by choices of bases as in (4)

$$\{ \underset{|}{\gamma}^1,\ldots,\underset{|}{\gamma}^4 \},\ \{ \overset{|}{\delta}_1,\ldots,\overset{|}{\delta}_4 \} = \{ \underset{|}{A},\underset{|}{B},\underset{|}{Y},\underset{|}{\gamma}^4 \},\ \{ \overset{|}{C},\overset{|}{D},\overset{|}{X},\overset{|}{\delta}_4 \}\ ;\ \begin{matrix}\gamma^i \\ | \\ \delta_j\end{matrix} = \delta^i_j \quad \text{for } \max\{i,j\} > 3\ ; \qquad (14)$$

and thus B_0 is completely defined.

Given the trivial embedding

$$\mathbb{C}^4_Z \times \mathbb{C}^{4*}_W \times \mathbb{C}^4_X \times \mathbb{C}^{4*}_Y \hookrightarrow \mathbb{C}^{4+n}_Z \times \mathbb{C}^{4+n*}_W \times \mathbb{C}^{4+n}_X \times \mathbb{C}^{4+n*}_Y \qquad (15)$$

we now construct a contour $B_n \hookleftarrow B_0$ in this enlarged space. We first define an "inflated" base $\pi_n B_n \sim C_{4+n,2}$ in the manner of (5) by extending the bases (11) subject to the same (extended) condition (12) where we can arrange $\mu(>4) = 1$. Then (13) continues to be satisfied for all $(\overset{|}{X},\underset{|}{Y}) \in \pi_n B_n$ and, by extensions of (14), we can equally define "inflated" fibres $F_{n\,(\overset{|}{X},\underset{|}{Y})}$ over $\pi_n B_n$ with

$$F_{0\,(\overset{|}{X},\underset{|}{Y})} \sim C_{4,3} \hookrightarrow F_{n\,(\overset{|}{X},\underset{|}{Y})} \sim C_{4+n,3} \quad \big((\overset{|}{X},\underset{|}{Y}) \in \pi_0 B_0 \hookrightarrow \pi_n B_n\big) \quad (16).$$

If (9) holds exactly one has

$$\left\langle \underset{|}{\alpha}^{3},\dots,\underset{|}{\alpha}^{n+4} \right\rangle = \left\langle \frac{\overset{|\,|\,|}{ABY}}{\underset{|\,|\,|}{CD}} = \frac{\overset{|\,|\,|}{EFY}}{\underset{|\,|\,|}{GH}}, \underset{|}{\gamma}^{3},\dots,\underset{|}{\gamma}^{n+4} \right\rangle \quad \text{and}$$

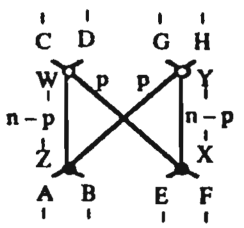
$$\left\langle \overset{|}{\beta}_{3},\dots,\overset{|}{\beta}_{n+4} \right\rangle = \left\langle \frac{\overset{|\,|\,|}{AB}}{\underset{|\,|\,|}{CDX}} = \frac{\overset{|\,|\,|}{EF}}{\underset{|\,|\,|}{GHX}}, \overset{|}{\delta}_{4},\dots,\overset{|}{\delta}_{n+4} \right\rangle \qquad (17)$$

and a restriction to extensions which are orthonormal in the sense

$$\frac{\gamma^{i}}{\bar{\gamma}_{j}} = \delta^{i}_{j} = \frac{\bar{\delta}^{i}}{\delta_{j}} , \quad \text{where} \quad \overline{z\underset{|}{\alpha}^{i}} = \bar{z}\overset{|}{\beta}_{i} \quad \text{for} \quad z \in \mathbb{C}\,;\; i,j = 3,\dots,n+4,$$

defines $F_{n}(\overset{|}{X},\underset{|}{Y})$ uniquely. The general case with (9) follows from continuity.

Thus we get a contour B_n over which we can integrate the forms



$$\longleftrightarrow \quad \omega_{n-p,p} = \frac{[(n-p)!\,p!]^{2}\, D^{4+n}Z\, D^{4+n}W\, D^{4+n}X\, D^{4+n}Y}{\begin{pmatrix} Y \\ | \\ Z \end{pmatrix}^{p+1} \frac{AB}{ZZ}\begin{pmatrix} W \\ | \\ Z \end{pmatrix}^{n-p+1} \frac{WW}{CD}\begin{pmatrix} W \\ | \\ X \end{pmatrix}^{p+1} \frac{EF}{XX}\begin{pmatrix} Y \\ | \\ X \end{pmatrix}^{n-p+1} \frac{YY}{GH}} \cdot \qquad (18)$$

Integrating over the fibres $F_{n}(\overset{|}{X},\underset{|}{Y})$ we get

$$\int_{B_n} \omega_{n-p,p} = (2\pi i)^{n+8} p!\,(n-p)! \left(-\frac{\overset{|\,|}{AB}}{\underset{|\,|}{CD}}\right)^{p} \iint_{\pi_n B_n} \left[\left(\frac{\overset{|\,|\,|}{ABY}}{\underset{|\,|\,|}{CDX}}\right)^{p+1} \begin{pmatrix} Y \\ | \\ X \end{pmatrix}^{n-p+1} \frac{EFYY}{XXGH} \right]^{-1} D^{4+n}X\, D^{4+n}Y$$

and after integrating out $(S^1)^4$ around the exterior poles $\Sigma_i E_iX^i$, F_iX^i, Y_iG^i, $Y_iH^i = 0$ we are left with

$$(2\pi i)^{n+12}(-1)^{p} p!\,(n-p)!\, a^{-1} \int_{S^1\times S^{3+2n}} \bigwedge_i dx^i \bigwedge_j dy_j \Big[\sum_k \mu(k)\, x^k y_k\Big]^{-p-1} \Big[\sum_h x^h y_h\Big]^{p-1-n}$$

$$(2 < i,j,k,h \le n+4\,,\ \mu(>4) = 1)$$

$$\overset{\text{if } n>0}{=} (2\pi i)^{2n+15} \frac{(-1)^{p} p!\,(n-p)!}{n!}\, n\, a^{-1} \int_0^1 dt \int_0^1 \frac{(1-s)^{n-1} s\, ds}{(1+\kappa(t)s)^{p+1}} \,, \quad \kappa(t) = \mu(3)t + \mu(4)(1-t) - 1$$

(19)

and one finds that

$$\frac{1}{(2\pi i)^{2n+15}} \int_{B_n} \sum_{p=0}^{n} (-1)^{p} \binom{n}{p} \omega_{n-p,p} = \frac{\lg \lambda_1/\lambda_2}{\lambda_1 - \lambda_2} \quad \text{independent of } n \ge 0! \qquad (20)$$

---

where $\lambda_i = a\,\mu(2+i)$, $i = 1,2$, are the roots of

$$\lambda^2 - \left(a + c - \begin{smallmatrix} A & B & E & F \\ | & | & | & | \\ | & | & | & | \\ C & D & G & H \end{smallmatrix}\right)\lambda + ac = 0\,. \qquad (21)$$

The significance of this result lies in the fact that the left hand side of (20) can be written as

$$\frac{1}{(2\pi i)^{2n+15}} \int_{B_n} \left[(\overset{\sqcap}{Z X} \partial_{\underset{\sqcup}{Z}} \partial_X \,/\, \overset{\sqcap}{ZX}\, \underset{\sqcup}{WY})^n\; K \right] F \; DZ\, DW\, DX\, DY \qquad (22)$$

with $K^{-1} = \begin{smallmatrix} W & W & Y & Y \\ | & | & | & | \\ Z & X & Z & X \end{smallmatrix}$, $F^{-1} = \begin{smallmatrix} A & B & E & F & W & W & Y & Y \\ | & | & | & | & | & | & | & | \\ Z & Z & X & X & C & D & G & H \end{smallmatrix}$

$\sqcup$, ($\sqcap$) is a fixed (hyper-)plane of complex (co-)dimension 2, generalising the infinity twistor. Thus if in a general scattering amplitude F contains a timelike propagator

$$(k_1 \cdot k_2)^{-1} \;\leftrightarrow\; (\overset{\sqcap}{ZX} \partial_{\underset{\sqcup}{Z}} \partial_X)^{-1} \quad , \quad F = (\overset{\sqcap}{ZX} \partial_{\underset{\sqcup}{Z}} \partial_X)^{-1}\, \overset{\sqcap}{ZX}\, \underset{\sqcup}{WY}\, \tilde{F} \qquad (23)$$

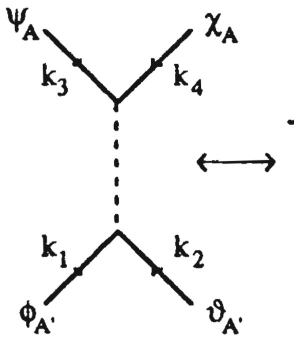
then integration by parts leads to the expression

$$\frac{1}{(2\pi i)^{2n+15}} \int_{B_n} \left[(\overset{\sqcap}{ZX} \partial_{\underset{\sqcup}{Z}} \partial_X \,/\, \overset{\sqcap}{ZX}\, \underset{\sqcup}{WY})^{n-1}\; K \right] \tilde{F} \quad , \quad n > 0 \qquad (24)$$

which in special examples can again be shown to be essentially independent of dimension.

Scattering amplitudes

As an example we consider the massless Yukawa process



$$\longleftrightarrow \int \Big[d^4k_1\, d^4k_2\, d^4k_3\, d^4k_4\, \delta(k_1 + k_2 + k_3 + k_4)\, \delta^-(k_4^2)\, \chi^A(k_4) \times$$

$$\delta^-(k_3^2)\, \psi_A(k_3) \frac{1}{(k_1 + k_2)^2}\, \delta^+(k_2^2)\, \vartheta_{A'}(k_2)\, \delta^+(k_1^2)\, \phi^{A'}(k_1) \Big] \qquad (25)$$

If we take special twistor representations for the exterior fields

$$\phi_{A'}(k_1) \leftrightarrow \overset{\sqcap}{Z} \begin{pmatrix} A \\ | \\ Z \end{pmatrix}^{-2} \begin{pmatrix} B \\ | \\ Z \end{pmatrix}^{-1} \qquad \psi_A(k_3) \leftrightarrow \underset{\sqcup}{W} \begin{pmatrix} W \\ | \\ C \end{pmatrix}^{-2} \begin{pmatrix} W \\ | \\ D \end{pmatrix}^{-1}$$

$$\vartheta_{A'}(k_2) \leftrightarrow \overset{\sqcap}{X} \begin{pmatrix} E \\ | \\ X \end{pmatrix}^{-2} \begin{pmatrix} F \\ | \\ X \end{pmatrix}^{-1} \qquad \chi_A(k_4) \leftrightarrow \underset{\sqcup}{Y} \begin{pmatrix} Y \\ | \\ G \end{pmatrix}^{-2} \begin{pmatrix} Y \\ | \\ H \end{pmatrix}^{-1} \qquad (26)$$

where "$\leftrightarrow$" is the usual Penrose correspondence with respect to some fixed $\mathbb{C}P^{3(*)} \subset \mathbb{C}P^{3+n(*)}$, then

$$F = (\overset{\sqcap}{ZX} \partial_{\underset{\sqcup}{Z}} \partial_X)^{-1}\, \overset{\sqcap}{ZX}\, \underset{\sqcup}{WY}\, \tilde{F} = (\overset{\sqcap}{ZX} \partial_{\underset{\sqcup}{Z}} \partial_X)^{-1}\, \overset{\sqcap}{ZX}\, \underset{\sqcup}{WY} \begin{pmatrix} A & E & W & Y \\ | & | & | & | \\ Z & X & C & G \end{pmatrix}^{-2} \begin{pmatrix} B & F & W & Y \\ | & | & | & | \\ Z & X & D & H \end{pmatrix}^{-1} \qquad (27)$$

and we can give the scattering amplitude in terms of higher dimensional single box twistor diagrams as

$$\frac{1}{(2\pi i)^{2n+15}} \int_{B_n} \left[(\underbrace{\partial_Z \partial_X} / \underbrace{WY})^{n-1} K\right] \begin{pmatrix} A & E & W & Y \\ | & | & | & | \\ Z & X & C & G \end{pmatrix}^{-2} \begin{pmatrix} B & F & W & Y \\ | & | & | & | \\ Z & X & D & H \end{pmatrix}^{-1} D^{4+n}Z\, D^{4+n}W\, D^{4+n}X\, D^{4+n}Y \tag{28}$$

This is in fact the same as

$$\frac{1}{n(n+1)} \left[\begin{matrix} \partial_C & \partial_G \\ | & | \\ \partial_A & \partial_B \end{matrix} - \begin{matrix} \partial_C & \partial_G \\ | & | \\ \partial_B & \partial_A \end{matrix} \right] \frac{1}{(2\pi i)^{15+2n}} \int_{B_n} \left[(\underbrace{\partial_Z \partial_X} / \underbrace{WY})^{n} K\right] \frac{D^{4+n}Z D^{4+n}W D^{4+n}X D^{4+n}Y}{\begin{matrix} A & E & W & Y & B & F & W & Y \\ | & | & | & | & | & | & | & | \\ Z & X & C & G & Z & X & D & H \end{matrix}} \tag{29}$$

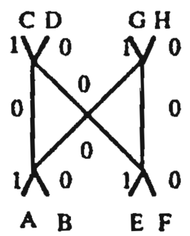
which can be calculated from (20)

$$\frac{1}{n(n+1)} \left[\begin{matrix} \partial_C & \partial_G \\ | & | \\ \partial_A & \partial_B \end{matrix} - \begin{matrix} \partial_C & \partial_G \\ | & | \\ \partial_B & \partial_A \end{matrix} \right] \frac{\lg \lambda_1/\lambda_2}{\lambda_1 - \lambda_2} = \left[\frac{\partial^2}{\begin{matrix} A & & B \\ \partial\,| & \partial & | \\ C & & G \end{matrix}} - \frac{\partial^2}{\begin{matrix} B & & A \\ \partial\,| & \partial & | \\ C & & G \end{matrix}} \right] \frac{\lg \lambda_1/\lambda_2}{\lambda_1 - \lambda_2} \tag{30}$$

where on the right λ_1, λ_2 are taken to be functions of

$$z^1, \ldots, z^{16} = \begin{matrix} A & A & A & A & B & B & B & B & E & E & E & E & F & F & F & F \\ |, & |, & |, & |, & |, & |, & |, & |, & |, & |, & |, & |, & |, & |, & |, & | \\ C & D & G & H & C & D & G & H & C & D & G & H & C & D & G & H \end{matrix} . \tag{31}$$

Thus we get an expression which is independent of the dimension n. It can be taken to define the left hand side in the case n = 0 which is of the form 0/0. For a twistor diagrammatic representation we take n=1:



(32)

<u>Remarks:</u>

1. There are also contours for the higher dimensional double box which allow representations of the right hand side of (20) involving space-like propagators and thus lead for example to a (dimension independent?) regularisation of Møller scattering.

2. It seems that a more invariant description of the contour B_n can be given as a bundle over the Grassmannian $Gr_2(\mathbb{C}^{2+n})$ with fibre B_0.

Franz Müller

The Quantum Potential – Possible Origins?

If we substitute $\psi = Re^{iS/\hbar}$ into Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

and define $\rho = R^2$, $\underline{v} = \text{grad}\, S/m$, then we obtain the equations

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \underline{v}) = 0 \qquad ①$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}.\nabla \underline{v} = -\text{grad}\left(\frac{V}{m} - \frac{\hbar^2}{2m^2}\frac{\nabla^2 R}{R}\right) \qquad ②$$

Because of the definition of $\underline{v}$ we should also append

$$\text{curl}\, \underline{v} = 0 \qquad ③$$

[Using ②, this need only be imposed as an initial value constraint.]

The <u>quantum potential</u> is the term in ② given by

$$Q(x,t) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \qquad ④$$

If Q is omitted, then ① & ② describe the motion of an ensemble of Newtonian particles with velocity field $\underline{v}(x,t)$, and particle density $\rho(x,t)$ moving in potential field V. The continuity equation ① guarantees particle conservation and ② is the equation of motion

$$\frac{d^2\underline{x}}{dt^2} = -\text{grad}\,(\text{potential}/m) \quad \text{for each particle path } \underline{x}(t).$$

One is therefore led to a "classical" interpretation of Schrödingers equation in terms of an ensemble of classical particles moving in an <u>extra</u> potential given by ④, and all quantum effects are attributable to this peculiar term. These ideas are the starting point for De Broglie's pilot wave theory and Bohm's causal interpretation of quantum mechanics. It is interesting to speculate on the origin of $Q(x,t)$ with its unusually high order and nonlinear form.

We give here three, essentially mathematical, remarks, in the hope that they may lead to some physical insights.

① Solving the Continuity Equation.

The continuity equation ① is unchanged by external potentials and may be viewed as a "kinematical" relation (rather like "$v = \frac{dx}{dt}$" in particle mechanics, thinking of ρ as a kind of "position" variable). We can write down the general solution of ① and substitute it into ②.

One way to do this is to write

$$j^\mu = (\rho, \rho \underline{v}) \quad \mu = 0,1,2,3 \quad \text{so ① is } \partial_\mu j^\mu = 0$$

Then the general solution may be written

$$j^\mu = \epsilon^{\mu\alpha\beta\gamma} \partial_\alpha \lambda_1 \partial_\beta \lambda_2 \partial_\gamma \lambda_3$$

where $\lambda_1(x,t)$, $\lambda_2(x,t)$, $\lambda_3(x,t)$ are arbitrary functions.

This gives

$$\rho = \det \Lambda \quad , \quad v_i = -[\Lambda^{-1}]_{ki} \frac{\partial \lambda_k}{\partial t} \quad , \text{ where } \Lambda_{ij} = \partial_i \lambda_j \qquad ⑤$$

The λ's have the following geometrical interpretation:

At $t=0$ choose coordinates $\tilde{\underline{x}} = \varphi(\underline{x})$ making the density 1. This requires φ chosen so that

$$\rho(x,0) = \mathrm{Jac}(\partial \varphi / \partial \underline{x})$$

Then set $\underline{\lambda}(\underline{x},0) = \underline{\varphi}(\underline{x})$ and evolve constantly along the integral curves of $\underline{v}$ to get $\lambda(x,t)$.

i.e. $\lambda(x,t) = \tilde{x}$ coordinate at $t=0$ of the particle at (x,t).

Substituting the λ's into ② gives some rather complicated terms, but a remarkable simplification occurs:

In one space dimension, $\rho = \lambda_x$, $v = -\lambda_t / \lambda_x$

and we get :

$$\frac{\partial v}{\partial t} + v.\nabla v = \left[-\frac{\lambda_{tt}}{\lambda_x} + \frac{2\lambda_t \lambda_{xt}}{\lambda_x^2} - \frac{\lambda_{xx}\lambda_t^2}{\lambda_x^3} \right] \quad (6)$$

$$\mathrm{grad}\left(\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right) = \frac{-\hbar^2}{4m}\left[-\frac{\lambda_{xxxx}}{\lambda_x} + \frac{2\lambda_{xx}\lambda_{xxx}}{\lambda_x^2} - \frac{\lambda_{xx}^3}{\lambda_x^3} \right] \quad (7)$$

Thus the quantum force term (7) arises by replacing t derivatives by xx derivatives in the convective derivative expression (6)! This curious "symmetry" is also seen in the rather simple Lagrangian which generates all the terms in (6) and (7):

$$L(\lambda_t, \lambda_x, \lambda_{xx}) = \frac{1}{2\lambda_x}\left[\lambda_t^2 - \left(\frac{\hbar^2}{4m^2}\right)\lambda_{xx}^2 \right] \quad (8)$$

In the 3-dimensional case, where the terms are far more complicated (and (3) becomes nontrivial) the "symmetry" still holds, if a slightly different form of the general solution of the continuity equation is used. (to be described elsewhere)

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(2) Curvature Interpretations.

The expression (4) for the quantum potential is similar to various curvature formulas in differential geometry.

$2\nabla^2 R/R$ is the scalar curvature $\mathcal{R}$ of the 4-metric

$$ds^2 = \rho\, dt^2 - (dx^2 + dy^2 + dz^2)$$

and the free Schrödinger equation becomes

$$\frac{\partial v}{\partial t} + \underline{v}.\nabla\underline{v} = \mathrm{grad}\left(\frac{\hbar^2}{2m^2}\mathcal{R}\right) \quad (9)$$

In a speculative mind, we may note that on the RHS, grad (scalar curvature) = div (2 Ricci tensor) [spatial components] and that the LHS of (9) is closely related to div ($T^{\mu\nu}$)

for $T^{\mu\nu} = \rho v^\mu v^\nu$, $v^\mu = (1, \underline{v})$, the stress energy tensor of a perfect fluid. This suggests that we might consider removing the div's from both sides to obtain an underlying tensorial equation, rather similar to Einsteins equations!

Santamato (in Phys. Rev. 1984) has related the quantum potential to the curvature of a Weyl geometry in 3 dimensions.

Consider the Weyl connection ∇ characterised by the 1form Q:

$$\nabla_c g^{ab} = Q_c g^{ab}$$

In 3-dimensions take $g_{ab} = \delta_{ab}$, $Q_a = 2\partial_a \ln \rho$

Then the scalar curvature $R_{ab} g^{ab}$ of the Weyl connection gives $\nabla^2 R/R$. Note that Q is exact here, so the Weyl connection is equivalent to the metric connection of a conformally rescaled g_{ab} ($= \rho^2 \delta_{ab}$)

3) a Gauge Theory?

In Schrödingers first paper on wave mechanics (~Ann. Phys. 1926) he obtains his equation from the Hamilton-Jacobi equation

$$H\left(q, \frac{\partial S}{\partial q}\right) = E \qquad p = \frac{\partial S}{\partial q}$$

by the replacement $S \rightsquigarrow k \ln \psi$, $k = i\hbar$.

This replacement (which generates the quantum potential term in the ρ-$\underline{v}$ equation) may be written as (with $R, S, \underline{v}$ as before)

$$\frac{im\underline{v}}{\hbar} \rightsquigarrow \frac{im\underline{v}}{\hbar} + R^{-1}\,\mathrm{grad}\,R \qquad (10)$$

which is similar to the gauge potential transformation law

$$A \rightsquigarrow g^{-1} A g + g^{-1} \partial_\mu g \qquad (11)$$

(for an Abelian group). If we now introduce the $\underset{\sim}{\lambda}$-variables of (5), then (10) reads

$$\underline{v} \rightsquigarrow \underline{v} - \frac{i\hbar}{2m} \text{grad} \ln \det \Lambda \qquad (12)$$

The special form of the last term suggests we use the facts:

1. $\text{grad} \ln \det A = \text{tr}(A^{-1} \text{grad} A)$
2. $\text{tr}(B^{-1} A B) = \text{tr} A$
3. Using (5), v_i can be naturally written as a trace of a 3x3 matrix $V^{(i)}$

$$v_i = \text{tr}(V^{(i)}) \qquad V^{(i)}_{jk} = [\Lambda^{-1}]_{ji} \frac{\partial \lambda_k}{\partial t}$$

Then (12) becomes

$$\text{tr}(V^{(i)}) \rightsquigarrow \text{tr}(\Lambda^{-1} V^{(i)} \Lambda) - \frac{i\hbar}{2m} \text{tr}(\Lambda^{-1} \text{grad} \Lambda) \qquad (13)$$

Again we may imagine removing all the traces (and comparing to (11)) we obtain the gauge transformation law for a larger underlying non abelian theory.

Richard Jozsa.

Update on one extended Regge trajectory

Of the various topics in the Twistor Particle Programme, I'd like to single out the extended Regge trajectories, in fact just one of them. I remind readers that these are straight line fits to various particle resonances, extended by a freedom or symmetry ϵ inherent in the definition of spin:

$$j(j+1) = \vec{J}^2 = J(J+1)$$

$$\Rightarrow \quad j + \tfrac{1}{2} = \epsilon(J + \tfrac{1}{2}), \quad \epsilon = \pm 1.$$

It was shown in 1976[1] that e.g. for the $I=0$ natural parity mesons we could fit them well on a straight line if we plot ϵm^2 against j (Fig. 1, details in Ref. 1). The 1990 Data booklet lists more resonances and following the scheme of Ref. 1) we get an even more interesting picture, Fig. 2.

A rough statistical analysis gives the following very encouraging result. Regressing on the leading resonances ω, f_2, ω_3 and f_4 gives the equation

$$y = 0.58 + 0.83\,x, \quad \text{standard deviation} = 0.12.$$

Regressing on the leading resonances plus f_0, ω, f_2 gives the extended trajectory:

$$y = 0.49 + 0.87x, \quad \text{standard deviation} = 0.18.$$

However, if we put the last three resonances on a "daughter" trajectory with conventional J and m^2, we get

$$y = -1.57 + 0.90x, \quad \text{standard deviation} = 0.35,$$

which shows it to be a worse fit. Also the slope is not universal.

More sophisticated analysis is being carried out.

Thanks to one of my co-authors in Ref. 1.

Reference: 1) Roger Penrose, George A J Sparling and Tsou Sheung Tsun, J. Phys. A 11 (1978) L231.

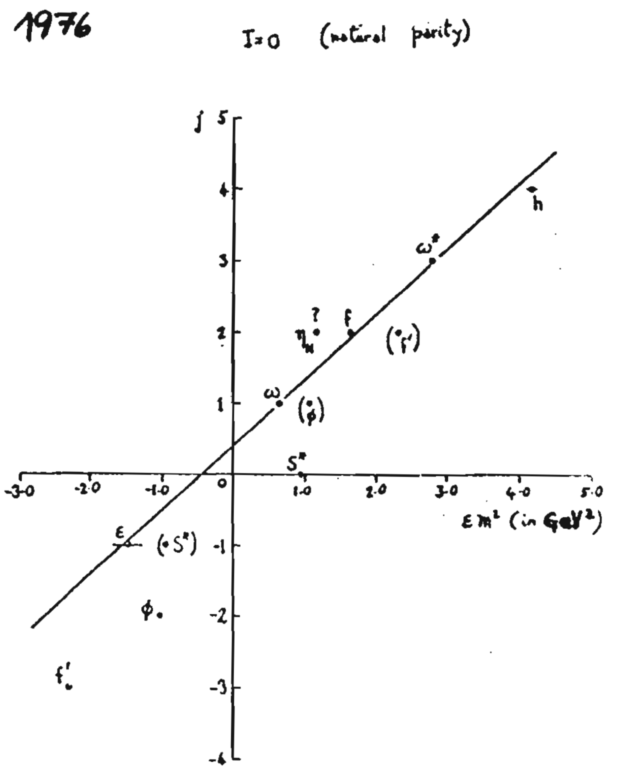

Fig. 1

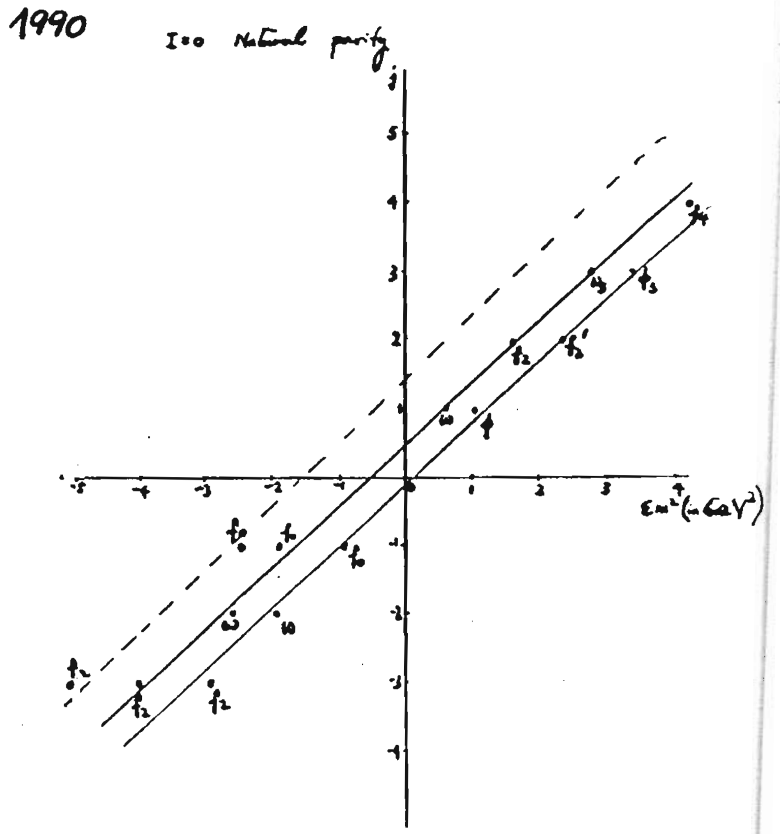

Fig. 2

Tsou Sheung Tsun

An inverse twistor function

We know from the exact sequence argument that, under very general conditions, a zero-rest-mass field on complex Minkowski space can be generated from a twistor function.[1] There are various ways of constructing the twistor function explicitly. The first was found by Brian Bramson, George Sparling, and Roger Penrose.[2] In this note, I shall describe a very simple construction that extends Richard Ward's treatment of the positive homogeneity case.[3] It must be 'known', since it is really no more than a concrete version of the exact sequence argument, but I have not been able to find it in print, so it may not be 'well-known'.

I shall deal only with the wave equation, leaving the general case as an easy exercise. Let $\phi(x^a)$ be a holomorphic solution of $\Box\phi = 0$ on complex Minkowski space and let $o^{A'}$, $\iota^{A'}$ be a constant spinor dyad, with $o_{A'}\iota^{A'} = 1$. Put

$$f(x^a, \pi_{A'}) = \frac{\phi(x^a)}{o^{A'}\pi_{A'}\iota^{B'}\pi_{B'}}.$$

This function is defined on the primed-spinor bundle, less the zero sets of $o^{A'}\pi_{A'}$ and $\iota^{A'}\pi_{A'}$. It is homogeneous of degree -2 in $\pi_{A'}$, and it generates ϕ in the sense that

$$\phi(x^a) = \frac{1}{2\pi \mathrm{i}} \oint f(x^a, \pi_{A'})\pi_{B'}\mathrm{d}\pi^{B'}.$$

The only problem is that it does not descend to twistor space since it is not constant along the vector fields $\pi^{A'}\nabla_{AA'}$, which span the fibration of the primed-spinor bundle over $\mathbf{T}$. In fact,

$$\pi^{A'}\nabla_{AA'}f = \frac{\pi^{A'}\nabla_{AA'}\phi}{o^{B'}\pi_{B'}\iota^{C'}\pi_{C'}} = -\frac{\iota^{A'}\nabla_{AA'}\phi}{\iota^{B'}\pi_{B'}} + \frac{o^{A'}\nabla_{AA'}\phi}{o^{B'}\pi_{B'}}.$$

Let us look for holomorphic functions g_0 and g_1 on the complements in the primed-spinor bundle of $o^{A'}\pi_{A'} = 0$ and $\iota^{A'}\pi_{A'} = 0$, respectively, such that

$$\pi^{A'}\nabla_{AA'}g_0 = \frac{o^{A'}\nabla_{AA'}\phi}{o^{B'}\pi_{B'}} \quad \text{and} \quad \pi^{A'}\nabla_{AA'}g_1 = \frac{\iota^{A'}\nabla_{AA'}\phi}{\iota^{B'}\pi_{B'}}.$$

Then $f - g_0 + g_1$ will also generate ϕ, and will be constant along the fibration, and so will be a twistor function for ϕ.

The integrability conditions for the existence of g_0 and g_1 are satisfied as a consequence of the wave equation. In fact, if we can find a surface S_1 in complex Minkowski space that intersects each α-plane in the domain of ϕ in a single point, with the exception of the α-planes tangent to $\iota_{A'}$, then we can take

$$g_1(x^a, \pi_{A'}) = \int_x^y \frac{\iota^{A'}\iota_{B'}\nabla_{AA'}\phi \, \mathrm{d}x^{AB'}}{(\iota^{C'}\pi_{C'})^2},$$

where y is the intersection point of S_1 with the α-plane through x tangent to $\pi_{A'}$ and the integral is along any path in this α-plane from x to y. With $\pi_{A'}$ fixed, the integrand is closed, so the choice of path is immaterial (under the obvious topological conditions). We can similarly define g_0 by first choosing S_0 to intersect the α-planes that are not tangent to $o_{A'}$. (The requirements on S_0 and S_1 have been stated in a more restrictive form than is necessary: all we need is that g_0 and g_1 should be defined when $\pi_{A'}$ lies in appropriate neighbourhoods of $\iota_{A'}$ and $o_{A'}$, respectively.)

One possibility is to take S_0 to be a fixed α-plane parallel to $o_{A'}$ and S_1 to be a fixed α-plane parallel to $\iota_{A'}$. Then by taking x to lie on S_0, we can construct an explicit twistor function F as follows: let Z be an α-plane with tangent spinor $\pi_{A'}$ such that $o^{A'}\pi_{A'} \neq 0 \neq \iota^{A'}\pi_{A'}$ and put

$$F(Z) = \frac{\phi(x)}{o^{A'}\pi_{A'}\iota^{B'}\pi_{B'}} + \int_x^y \frac{\iota^{A'}\iota_{B'}\nabla_{AA'}\phi \, \mathrm{d}x^{AB'}}{(\iota^{C'}\pi_{C'})^2},$$

where x is the intersection of Z with S_0 and y is the intersection of Z with S_1.

Example. Let h be a holomorphic function of a single (unprimed) spinor variable. Then $\phi(x) = h(x^{AA'}\iota_{A'})$ is a solution of the wave equation. In this case, the integral term in the definition of F vanishes, leaving

$$F(Z) = \frac{\phi(x)}{o^{A'}\pi_{A'}\iota^{B'}\pi_{B'}},$$

where x is the intersection point of Z with S_0. If we take S_0 to be the α-plane through the origin tangent to $o^{A'}$, then $x^{AA'} = -\mathrm{i}\omega^A o^{A'}/o^{B'}\pi_{B'}$, where $Z = (\omega^A, \pi_{A'})$. A simple application of Cauchy's theorem verifies that

$$F(\omega^A, \pi_{A'}) = \frac{h(\mathrm{i}\omega^A/o^{B'}\pi_{B'})}{o^{C'}\pi_{C'}\iota^{D'}\pi_{D'}}$$

is indeed a twistor function for ϕ.

Nick Woodhouse

References

1. M. G. Eastwood, R. Penrose, and R. O. Wells Jr: Cohomology and massless fields, *Commun. Math. Phys.* **78**, 305-51 (1981). R. S. Ward and R. O. Wells Jr: *Twistor geometry and field theory*, Cambridge University Press, Cambridge, 1990.

2. R. Penrose: Twistor theory, its aims and achievements, in *Quantum gravity, an Oxford symposium*, eds C. J. Isham, R. Penrose, and D. W. Sciama, Oxford University Press, Oxford, 1975.

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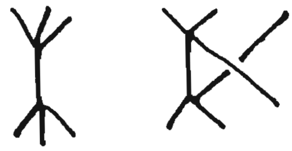
Update on Higher Order Feynman Diagrams

I can now give a much improved analysis of the second-order ϕ^4 integral discussed in **TN** 31, based on new formulas for integrating the Feynman propagator. This eliminates the guessing in that TN, but unfortunately shows my guess was wrong. I have to abandon the claim that the ultra violet divergent loop diagram

can be derived immediately from the tree diagrams



However, the general line of analysis is still valid, and the new formulas considerably extend the scope for higher-order calculations and take us nearer a transcription of general Feynman diagrams.

The basic idea is that for twistor diagram translation we need to represent Δ_F as sandwiched between test fields, i.e. to consider

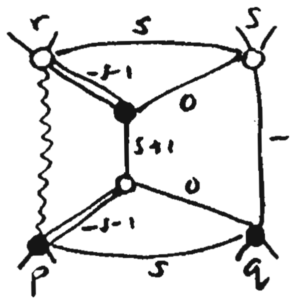
$$\int d^4x\, d^4y \;\; \phi_1(x)\, \Delta_F(x-y;\, m)\, \phi_2(y) \qquad (1)$$

where ϕ_1, ϕ_2 are unconstrained fields. It's very helpful to break this down into the cases where the test fields ϕ_i are *timelike* (i.e. positive or negative frequency) or *spacelike*. Then in the timelike case it's sufficient to consider

$$\int d^4x\, d^4y \; \frac{1}{(x-p)^2(x-q)^2} \; \Delta_F(x-y;\, m) \; \frac{1}{(y-r)^2(y-s)^2} \qquad (2)$$

where $\hat{p}, \hat{q}$, are in the past tube; $\hat{r}, \hat{s}$ in the future tube. This integral was evaluated by fairly elementary methods in my thesis long ago (1975) but I wasn't able then to see a twistor-diagram-like integral to represent it. More recent work shows that one can in fact be given as

$$\int ds \; \frac{(m^2)^s}{\sin \pi s}$$



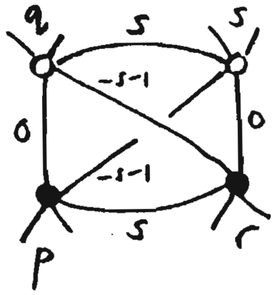
(3)

where the s-integral is a Barnes integral like this:

(The proof of this and other formulas are too involved for this brief note.)

Now consider the spacelike case, for which it's sufficient again to consider the expression (2), but now with $p^{\hat{}}, r^{\hat{}}$ in the past tube and $q^{\hat{}}, s^{\hat{}}$ in the future tube. In this case I *was* able in 1975 to give the twistor-diagram-like integral

$$\int ds \, \frac{(m^2)^s}{\sin \pi s}$$

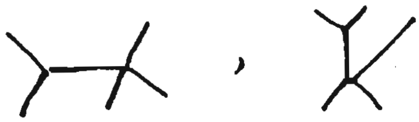


(4)

but wasn't able to prove it correct. This has now been done. (It's a pity I didn't push my 1975 line of thought to this conclusion a long time ago.)

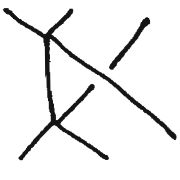
As a corollary, twistor diagrams for all first-order massless scattering amplitudes can be deduced via appropriate spin-raising and the limit $m \to 0$.

The formulas can also be applied to evaluate the Feynman diagrams



etc. as discussed by me in **TN** 29 and TMP, by me and L.J.O'D. in **TN** 30, and used to verify an exact correspondence with twistor diagrams.

However, the payoff I want to describe here is how they can be applied to the second-order ϕ^4 diagram



i.e. to the calculation of

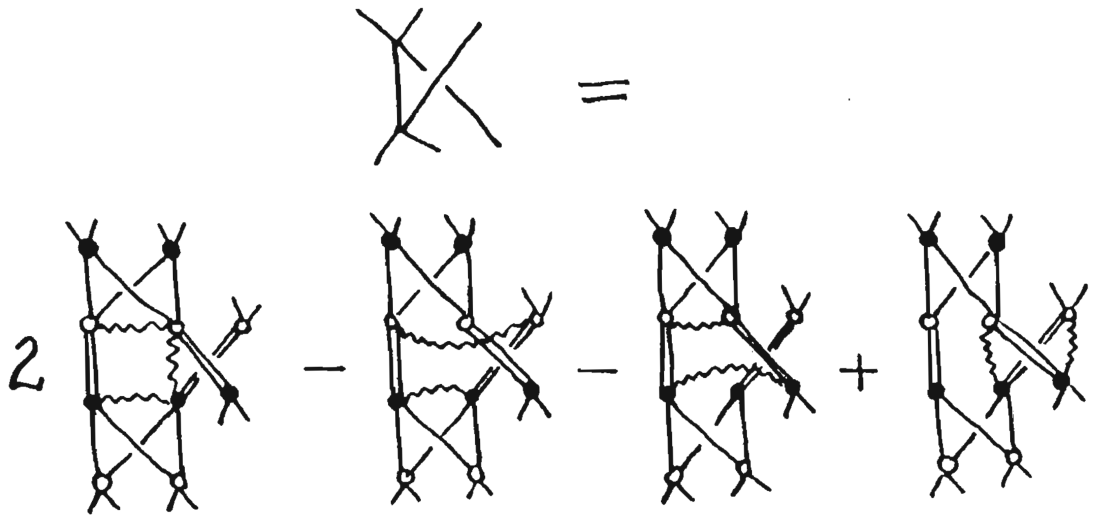
$$\int d^4x\, d^4y \; \frac{1}{[(x-p)^2]^2 (x-s)^2} \; \Delta_F(x-y;0) \; \frac{1}{[(y-q)^2]^2 (y-r)^2}$$

where $p^{\hat{}}, r^{\hat{}}$ are in the past tube; $q^{\hat{}}, s^{\hat{}}$ in the future tube.

This is straightforward once we observe that

$$\frac{1}{[(x-p)^2]^2 (x-s)^2} = \frac{1}{[(x-p)^2]^2 (p-s)^2} + \frac{\partial}{\partial p} \cdot \frac{\partial}{\partial p} \frac{\log((p-s)^2)}{(x-s)^2 (x-p)^2}$$

where the first term is manifestly timelike and the second term manifestly spacelike. Considering these two parts separately, the formulas (3) and (4) induce expressions for the two parts respectively. These expressions can in turn be represented as inhomogeneous twistor diagrams, all agreeing with the "skeleton" picture. More precisely, this must be done for a Feynman propagator with mass m, since we find that each part separately diverges as m $\to$ 0, although the sum of the two parts is convergent in this limit. The final result is that



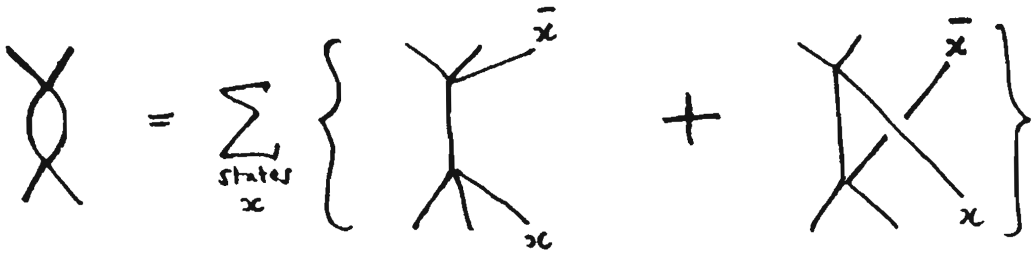
(5)

where although a mass m has to appear in the inhomogeneous boundaries at infinity, the sum of the diagrams is actually independent of m.

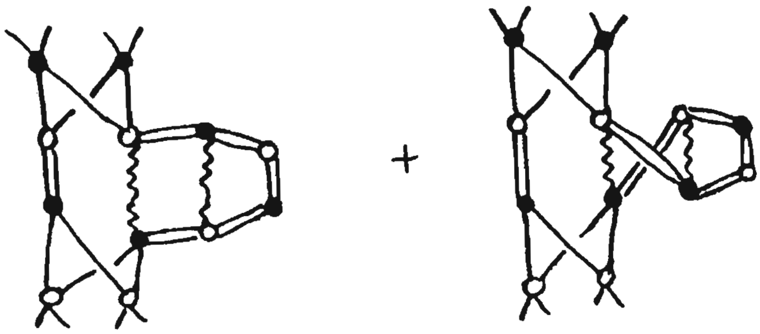
The techniques employed here could certainly be applied to other Feynman integrals. As an example, I have also computed an expression for the remaining channel

for this diagram. It appears that this too can be represented by a twistor diagram with the expected "skeleton". However, in view of the fact that (3) and (4) actually capture all the content of the Feynman propagator, it should be possible to extend these techniques much further, at least to tree diagrams.

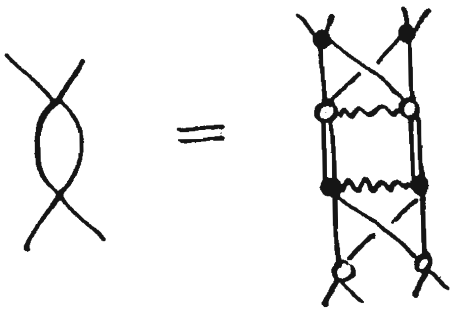
Finally there is the question of deriving the loop diagram from this result. When we now sum over external states to make the loop, as described in **TN** 31, the first term in (5) makes a contribution which is exactly cancelled by the second and third terms. Thus



is formally



In my previous note I thought that these terms would cancel each other but this now seems to be wrong. Instead, new analysis suggests that to make sense of the divergent summation, further inhomogeneous boundaries at infinity must be added, and that then we do obtain as required,



Considerable work has been done by L.J. O'D. on this calculation, developing ideas suggested in **TN** 32 about the essential role of inhomogeneity in twistor diagrams.

Andrew Hodges

TWISTOR NEWSLETTER No. 33

Contents

* Papers given at the birthday seminar for Roger Penrose

Short contributions for TN 34 should be sent to

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