# The most general {22} self-dual vacuum: a googly approach

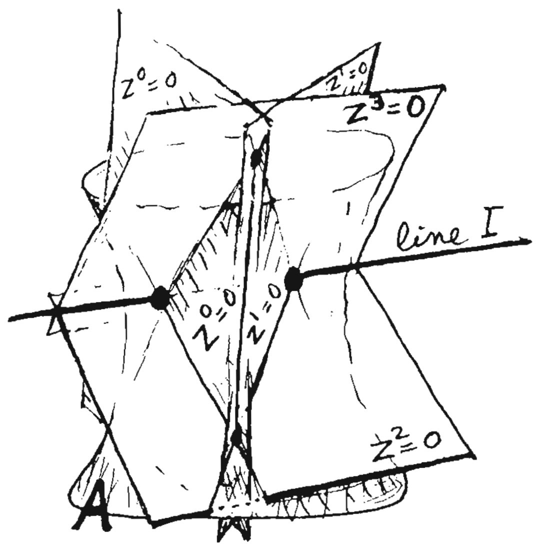
Twistor functions f of homogeneity degree −6 can be used to generate curved dual twistor spaces and, thence, self-dual (complex) vacuum "space-times" $\mathcal{M}$. There are some plausible advantages in proceeding in this "googly" fashion (cf. R.P. in TN 16,17,23). Any symmetry that the twistor function f possesses will be carried over to a symmetry of $\mathcal{M}$, and there is a natural specialization to the linearized limit. Moreover, since in linear theory a pole-multiplicity $m$ in the −6 homogeneity function f is reflected in the (linearized) Weyl curvature possessing a $(5-m)$-fold principal null direction (cf. R.P., J. Math. Phys. 10 (1969) 38-9), we may anticipate something similar occurring in the fully non-linear self-dual case. We have been able to carry through this programme explicitly in the case when f is the inverse cube of a generic quadratic form, and find that, indeed, the resulting self-dual $\mathcal{M}$ has {2,2} Weyl tensor, being the self-dual specialization of the Plebański–Demiański (Annals of Physics 98 (1976) 98–127) solution, which is itself a NUT-type generalization of Levi-Civita's C-metric. (Our solution belongs to the class described by Tod and Ward, Proc. Roy. Soc. Lond. A 368 (1979) 411–427, and also one by Fette, Janis and Newman, J. Math. Phys. 17 (1976) 660.)

The general quadratic form $A_{\alpha\beta}Z^\alpha Z^\beta$, in the twistor variable $Z^\alpha$ can be put in the form

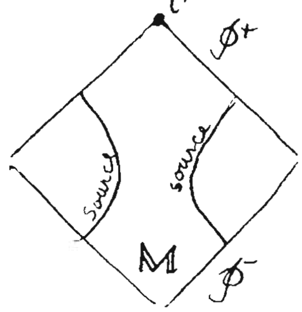
$$A_{\alpha\beta}Z^\alpha Z^\beta = Z^0 Z^1 + a^2 Z^2 Z^3$$
$$= \omega^0\omega^1 + a^2\pi_{0'}\pi_{1'} \qquad = \tfrac{1}{2}(Z^0 \ldots Z^3)\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & a \\ & & a & \end{pmatrix}\begin{pmatrix} Z^0 \\ \vdots \\ Z^3 \end{pmatrix}$$

by a suitable choice of twistor coordinates ($Z^0=\omega^0, Z^1=\omega^1, Z^2=\pi_{0'}, Z^3=\pi_{1'}$) and our twistor function can be taken as

$$f = \left(\frac{k}{A_{\alpha\beta}Z^\alpha Z^\beta}\right)^3 = \frac{k^3}{(Z^0Z^1 + a^2 Z^2 Z^3)^3}.$$




This function has just 3-fold poles lying along the quadric **A** given by $A_{\alpha\beta}Z^\alpha Z^\beta = 0$. The line **I** is given by $Z^2 = Z^3 = 0$, and we have taken the generic case whereby **I** meets **A** in two distinct points. If we put f into a twistor integral, we derive the linearized Weyl spinor of a linearized self-dual C-metric (uniformly accelerating NUT-like source). We have included



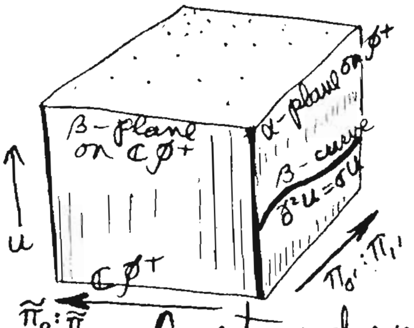

"a" as an explicit parameter so as to allow for the limit $a \to 0$, which gives an elementary state in linear theory and the Eguchi-Hanson-Sparling-Tod solution in the full theory ($\left.\begin{matrix}\text{K.P.T.}\\ \text{G.B.-S.}\\ \text{R.P.}\end{matrix}\right\}$ in TN 9).

We do not recover the linearized or full Schwarzschild-NUT specialization with our particular coordinate choice, according to which the line I would lie <u>on</u> A and the twistor function could be written in the form $k(Z^0Z^2 + Z^1Z^3)^{-3}$. (This solution might encounter difficulties with a googly-type approach in any case since its $\mathbb{C}\mathscr{I}^+$ "looks" flat.) We do recover <u>flat</u> space, however, taking $k \to 0$.

The procedure is first to derive a "Bondi" $\sigma$ $(=\sigma^\circ)$ on $\mathbb{C}\mathscr{I}^+$ (identified with the standard $\mathbb{C}\mathscr{I}^+$ for $\mathbb{C}M$ by matching the two at $i^+$) according to the formula

$$\frac{\partial^2\sigma}{\partial\omega^A\partial\omega^B} = \oint \tilde{\pi}_A \tilde{\pi}_B \, f(\omega^A + \lambda\tilde{\pi}^A, \pi_{A'})\, d\lambda .$$

We then solve the Newman equation "$\eth^2 u = \sigma u$" to derive the dual twistor lines ($\beta$-curves) on $\mathbb{C}\mathscr{I}^+$. The space of these lines provides us with a curved **dual** projective




twistor space $\mathbb{P}\mathcal{T}^*$, and we find patching relations for $\mathcal{T}^*$ explicitly. Holomorphic cross-sections of $\mathcal{T}^*$, i.e. "good cuts" of $\mathbb{C}\mathcal{I}^+$, give us the points of $\mathcal{M}$, and these can also be found explicitly.

A standard $u$-coordinate for $\mathbb{C}\mathcal{I}^+$ for $\mathbb{M}$ can be given as

$$u = i\tilde{\omega}^{A'}\pi_{A'} = -i\tilde{\pi}_A\omega^A$$

(where the flat space twistor $(\omega^A, \pi_{A'})$ and dual twistor $(\tilde{\pi}_A, \tilde{\omega}^{A'})$ are incident and define a null ray in $\mathbb{CM}$ meeting $\mathbb{C}\mathcal{I}^+$ in a point with coordinate $u$). We can specialize either to $\pi_{0'} = 1$ or $\pi_{1'} = 1$ and either to $\tilde{\pi}_0 = 1$ or $\tilde{\pi}_1 = 1$, if desired, but we prefer to consider $u$ as a homogeneity $(1,1)$ weighted expression and define a $(0,0)$ quantity $\rho$ by

$$\frac{u^2}{\pi_{0'}\pi_{1'}\tilde{\pi}_0\tilde{\pi}_1} = \frac{1}{\rho}(\rho + a^2)^2.$$

Note that $u$ is invariant under $\rho \mapsto a^4/\rho$.

It turns out that with our choice of $f$, the solution of Newman's equation for the $\beta$-curves on $\mathbb{C}\mathcal{I}^+$ is given by

$$F\,\frac{\pi_{1'}}{\pi_{0'}} = (\varpi + P)\left(\frac{\varpi + Q}{\varpi - Q}\right)^{a^2/Q}$$

— or the same with $\pi_{0'}$ and $\pi_{1'}$ interchanged — where $\varpi$ is related to $\rho$ by

$$\rho = \frac{\varpi^2 - Q^2}{2(\varpi + P)},$$

$Q$ being a constant

$$Q^2 = a^4 - 8k^3,$$

and $F$ and $P$ being functions of $\tilde{\pi}_0, \tilde{\pi}_1$ (weights $(0,0)$ each).

We have $\rho$ unchanged under

$$\varpi \mapsto \hat{\varpi} = -\frac{P\varpi + Q^2}{\varpi + P}.$$

This leads to the patching relation

$$\boxed{\hat{W}_0 = W_0,\quad \hat{W}_1 = W_1,\quad \hat{W}_2 = \left(1 + Q\frac{W_0W_1}{W_2W_3}\right)^{\frac{1}{2}\left(1-\frac{a^2}{Q}\right)} W_2,\quad \hat{W}_3 = \left(1 + Q\frac{W_0W_1}{W_2W_3}\right)^{\frac{1}{2}\left(\frac{a^2}{Q}-1\right)} W_3}$$

where
$$W_0 = \tilde{\pi}_0, \quad W_1 = \tilde{\pi}_1, \quad P - Q = \frac{2 W_2 W_3}{W_0 W_1}, \quad \frac{W_2}{W_3} = \frac{Q-P}{F}, \quad \frac{\hat{W}_2}{\hat{W}_3} = \frac{\hat{F}}{Q-P}$$
and
$$\hat{P} = P, \quad \hat{F} = (P^2 - Q^2)\left(\frac{P+Q}{P+Q}\right)^{a^2/Q} \cdot \frac{1}{F}$$
(opposite choice of $\pi_0'$, $\pi_1'$ in $\beta$-curve equation for $\hat{F}$). The patching relation between $\hat{W}_{\underset{\sim}{\alpha}}$ and $W_{\underset{\sim}{\alpha}}$ defines our required dual twistor space $\mathcal{T}^*$. We have $d\hat{W}_2 \wedge d\hat{W}_3 = dW_2 \wedge dW_2$ on constant $W_0, W_1$, as required.

Note that $\hat{W}_2 \hat{W}_3 = W_2 W_3$, and that the two commuting symmetries
$$W_0 \mapsto \alpha W_0, \; W_1 \mapsto \alpha^{-1} W_1 \quad \text{and} \quad W_2 \mapsto \beta W_2, \; W_3 \mapsto \beta^{-1} W_3$$
hold for $\mathcal{T}^*$ and therefore give two commuting Killing vector symmetries for $\mathcal{M}$. The holomorphic cross-sections for $\mathcal{T}^*$ can be found explicitly. The Weyl curvature for $\mathcal{M}$ is indeed self-dual $\{22\}$, the solution being of Plebanski-Demianski type.

It is of interest to examine the weak field limit, where $Q = a^2 + \varepsilon$ ($\varepsilon$ small). We find
$$\hat{W}_2 = W_2\left(1 + \frac{\varepsilon}{2a^2}\log\left(1 + a^2\frac{W_0 W_1}{W_2 W_3}\right)\right)$$
$$\hat{W}_3 = W_3\left(1 - \frac{\varepsilon}{2a^2}\log\left(1 + a^2\frac{W_0 W_1}{W_2 W_3}\right)\right)$$
with $\hat{W}_0 = W_0$, $\hat{W}_1 = W_1$. This comes from the standard formula
$$\hat{W}_\alpha = W_\alpha + \varepsilon I_{\alpha\beta} \frac{\partial}{\partial W_\beta} \tilde{f}(W)$$
where
$$2\tilde{f}(W) = B \log B - C \log C - D \log D$$
with
$$B = W_0 W_1 + \frac{1}{a^2} W_2 W_3, \quad C = \frac{W_2 W_3}{a^2}, \quad D = W_0 W_1,$$
so $B = C + D$, ensuring $+2$ homogeneity. The term $B \log B$ is the kind of thing one expects from twistor transform considerations in relation to our original $f(z)$, since $B^{\alpha\beta}$ is the inverse of $A_{\alpha\beta}$, where $B = B^{\alpha\beta} W_\alpha W_\beta$.

Linda Hackehunt & Roger Penrose