A comment on the preceding article

In the preceding article, Linda Haslehurst and Roger Penrose derive the twistor space of a self-dual type $D$ vacuum metric. As they remark, their solution is one of a class constructed by Tod and Ward from solutions of Laplace's equation in three variables, and by following through the analysis in [1], one can find an explicit form of the metric by solving a linear splitting problem.

Since the solution has two commuting Killing vectors, it also has a 'Yang-Mills' twistor description [2]: the solution is encoded in a holomorphic vector bundle over a one-dimensional non-Hausdorff Riemann surface and can be recovered from its patching matrix $P$, a $2 \times 2$ matrix-valued holomorphic function of a single complex variable $w$ [3,4]. The purpose of this note is to describe the connection between the two twistor constructions. The example is of interest in the context of the Yang-Mills construction because it illustrates what seems to be a general property of type $D$ solutions, that their patching matrices are rational functions of a particularly simple form. Thomas von Schroeter has verified this by direct calculation in the case of the Lorentzian type $D$ metrics, which are known explicitly, but the geometric reason for the special rational form is not yet clear. This self-dual example is another instance of the phenomenon, and it is one in which the underlying geometry is rather easier to understand.

To fit with the conventions of [4], I shall consider the dual version of the solution, for which the twistor space is determined by the coordinate relations

$$
\hat{Z}^0 = \left(1 + \frac{Q}{w}\right)^{n/Q} Z^0, \quad \hat{Z}^1 = \left(1 + \frac{Q}{w}\right)^{-p/Q} Z^1, \quad \hat{Z}^2 = Z^2, \quad \hat{Z}^3 = Z^3,
$$

where $w = Z^0 Z^1 / Z^2 Z^3$ and $p = \frac{1}{2}(Q - \alpha^2)$. The surfaces of constant $w$ are the leaves of the foliation of $\mathbb{P}T$ spanned by the Killing vectors. The connection between the two constructions is made by considering the holomorphic tangent bundle of $T$. This is the pull-back to $T$ of the rank-4 bundle $E \to \mathbb{P}T$ that has local sections of the form

$$
A^a \frac{\partial}{\partial Z^a},
$$

where the $A$s are holomorphic functions of the $Z$s, homogeneous of degree zero: $E$ is the Ward transform of the anti-self-dual Yang-Mills connection that defines local twistor transport in space-time.
Let $L \to \mathbb{PT}$ be the line bundle with sections represented by holomorphic functions of degree one in $Z^\alpha$. Construct an open cover of $\mathbb{PT}$ by taking $V_0, V_1, V_2, V_3$ to be suitable neighbourhoods of $Z^0 = 0$, $Z^1 = 0$, $Z^2 = 0$, and $Z^3 = 0$, and trivialize $E \otimes L$ by the four frame fields defined on the respective neighbourhoods by

$$(V_0) \quad X, \frac{Z^0}{w} \frac{\partial}{\partial Z^0}, -Z^2 \frac{\partial}{\partial Z^1}, X - Z^3 \frac{\partial}{\partial Z^3}$$

$$(V_1) \quad \frac{\hat{Z}^1}{w} \frac{\partial}{\partial \hat{Z}^1}, X, X - \hat{Z}^2 \frac{\partial}{\partial Z^2}, -\hat{Z}^3 \frac{\partial}{\partial Z^3}$$

$$(V_2) \quad X, Z^0 \frac{\partial}{\partial Z^0}, -wZ^2 \frac{\partial}{\partial Z^2}, X - Z^3 \frac{\partial}{\partial Z^3}$$

$$(V_3) \quad \frac{\hat{Z}^1}{\hat{Z}^1}, X, X - \hat{Z}^2 \frac{\partial}{\partial \hat{Z}^2}, -w\hat{Z}^3 \frac{\partial}{\partial \hat{Z}^3}$$

where $X = -Z^0 \partial/\partial Z^0 + Z^1 \partial/\partial Z^1 = -\hat{Z}^0 \partial/\partial \hat{Z}^0 + \hat{Z}^1 \partial/\partial \hat{Z}^1$ (the generator one of the symmetries). The corresponding transition matrices are $P_{01} = \text{diag}(1, w^{-1}, w^{-1}, 1)$, $P_{31} = \text{diag}(w^{-1}, 1, 1, w^{-1})$, and

$$P_{01} = \begin{pmatrix} g(w) & 1 & wg(w) \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where $g(w) = (q + w)/w(Q + w)$, with $q = \frac{1}{2}(Q + a^2)$. We interpret $w$ as the coordinate on the non-Hausdorff reduced twistor space.

Note that the vectors parallel to $Z^\alpha \partial/\partial Z^\alpha$ span a line sub-bundle of $E$ isomorphic to $L^{-1}$, and that $F := E/L^{-1} = T\mathbb{PT} \otimes L^{-1}$.

The general anti-self-dual vacuum metric with two commuting orthogonally transitive Killing vectors is

$$ds^2 = f(dt - \omega d\theta)^2 + f^{-1}(dr^2 + dz^2 + r^2 d\theta^2) \quad (*)$$

where $\omega$ and $f$ are functions of $r$ and $z$ such that $f^2 \omega_z = r f_z$ and $f^2 \omega_r = -r f_z$. By considering local twistor transport in such a background, one can show that the bundle $E \otimes L$ always has transition matrices of this form, where in the general case $g(z) = f(\theta, z)^{-1}$. The top left-hand two-by-two block in $P_{01}$
is the patching matrix $P$, which characterizes the solution uniquely. We can immediately read off, therefore, that the type $D$ solution in the preceding article is given by (*), with

$$\frac{1}{f(0,z)} = \frac{q}{Qz} + \frac{p}{Q(Q+z)}.$$  

Since $f^{-1}(r,z)$ is an axisymmetric harmonic function in cylindrical polar coordinates, we conclude that it is the potential of a pair of point masses $p/Q$ and $q/Q$ separated by $Q$.

A vacuum solution of the form (*) is type $D$ if it admits a non-null Killing spinor $\omega^{AB}$ (this is a nontrivial condition, although there are many Killing spinors with primed indices whatever the form of the curvature). A Killing spinor with unprimed indices determines a holomorphic section $A$ of the symmetric tensor product $F \otimes_S F = L^{-2} \otimes (TPT \otimes_S TPT)$. By dropping the fourth element of the local frame for $E$ in $V_0$ and $V_2$, and the third element in $V_1$ and $V_3$, we can represent $L \otimes F = TPT$ by the transition matrices $M_{20} = \text{diag}(1, w^{-1}, w^{-1})$, $M_{31} = \text{diag}(w^{-1}, 1, w^{-1})$, and

$$M_{01} = \begin{pmatrix} g(w) & 1 & -wg(w) \\ 1 & 0 & -2w \\ 0 & 0 & 1 \end{pmatrix}.$$  

A section of $F \otimes_S F$ has local representatives $A_i(w)$, where $i = 0, 1, 2, 3$ and the $A$s are symmetric $3 \times 3$ matrices. For a global section

$$A_2 = wM_{20}A_0M_{20}^t, \quad A_3 = wM_{31}A_1M_{31}^t, \quad A_0 = M_{01}A_1M_{01}^t,$$

with $A_2$ and $A_3$ well behaved near $w = \infty$, and $A_0$ and $A_1$ well behaved near $w = 0$. The first two transition relations imply that the dependence of $A_0$ and $A_1$ on $w$ must be of the form

$$A_0 = \begin{pmatrix} 0 & C & C \\ C & L & L \\ C & L & L \end{pmatrix}, \quad A_1 = \begin{pmatrix} L & C & L \\ C & 0 & C \\ L & C & L \end{pmatrix}$$

where $C$ means 'constant' and $L$ means 'linear'. After a little algebra, the third relation then implies that $g$ must be of the form $g = g_1(w)/g_2(w)$, where $g_1$ is at most linear in $w$ and $g_2$ is at most quadratic. For the solution in the previous article, $g_1 = q - pw$ and $g_2 = Qw + w^2$: this is essentially the generic case. For the 'Euclidean Taub-NUT' solution, $g_1 = w$ and $g_2 = w^2 + 2m$. 
