A comment on the preceding article

In the preceding article, Linda Haslehurst and Roger Penrose derive the twistor space of a self-dual type D vacuum metric. As they remark, their solution is one of a class constructed by Tod and Ward from solutions of Laplace's equation in three variables, and by following through the analysis in [1], one can find an explicit form of the metric by solving a linear splitting problem.

Since the solution has two commuting Killing vectors, it also has a 'Yang-Mills' twistor description [2]: the solution is encoded in a holomorphic vector bundle over a one-dimensional non-Hausdorff Riemann surface and can be recovered from its patching matrix P, a 2×2 matrix-valued holomorphic function of a single complex variable w [3,4]. The purpose of this note is to describe the connection between the two twistor constructions. The example is of interest in the context of the Yang-Mills construction because it illustrates what seems to be a general property of type D solutions, that their patching matrices are rational functions of a particularly simple form. Thomas von Schroeter has verified this by direct calculation in the case of the Lorentzian type D metrics, which are known explicitly, but the geometric reason for the special rational form is not yet clear. This self-dual example is another instance of the phenomenon, and it is one in which the underlying geometry is rather easier to understand.

To fit with the conventions of [4], I shall consider the dual version of the solution, for which the twistor space is determined by the coordinate relations

$$\hat{Z}^0 = \left(1 + \frac{Q}{w}\right)^{p/Q} Z^0, \; \hat{Z}^1 = \left(1 + \frac{Q}{w}\right)^{-p/Q} Z^1, \; \hat{Z}^2 = Z^2, \; \hat{Z}^3 = Z^3,$$

where $w = Z^0 Z^1 / Z^2 Z^3$ and $p = \frac{1}{2}(Q - a^2)$. The surfaces of constant w are the leaves of the foliation of $\mathbb{P}\mathcal{T}$ spanned by the Killing vectors. The connection between the two constructions is made by considering the holomorphic tangent bundle of $\mathcal{T}$. This is the pull-back to $\mathcal{T}$ of the rank-4 bundle $E \to \mathbb{P}\mathcal{T}$ that has local sections of the form

$$A^\alpha \frac{\partial}{\partial Z^\alpha},$$

where the As are holomorphic functions of the Zs, homogeneous of degree zero: E is the Ward transform of the anti-self-dual Yang-Mills connection that defines local twistor transport in space-time.

Let $L \to \mathbb{P}\mathcal{T}$ be the line bundle with sections represented by holomorphic functions of degree one in Z^α. Construct an open cover of $\mathbb{P}\mathcal{T}$ by taking V_0, V_1, V_2, V_3 to be suitable neighbourhoods of $Z^0 = 0$, $\hat{Z}^1 = 0$, $Z^2 = 0$, and $\hat{Z}^3 = 0$, and trivialize $E \otimes L$ by the four frame fields defined on the respective neighbourhoods by

$$
\begin{array}{ll}
(V_0) & X,\ \dfrac{Z^0}{w}\dfrac{\partial}{\partial Z^0},\ -Z^2\dfrac{\partial}{\partial Z^2},\ X - Z^3\dfrac{\partial}{\partial Z^3} \\
(V_1) & \dfrac{\hat{Z}^1}{w}\dfrac{\partial}{\partial \hat{Z}^1},\ X,\ X - \hat{Z}^2\dfrac{\partial}{\partial Z^2},\ -\hat{Z}^3\dfrac{\partial}{\partial \hat{Z}^3} \\
(V_2) & X,\ Z^0\dfrac{\partial}{\partial Z^0},\ -wZ^2\dfrac{\partial}{\partial Z^2},\ X - Z^3\dfrac{\partial}{\partial Z^3} \\
(V_3) & \hat{Z}^1\dfrac{\partial}{\partial \hat{Z}^1},\ X,\ X - \hat{Z}^2\dfrac{\partial}{\partial Z^2},\ -w\hat{Z}^3\dfrac{\partial}{\partial \hat{Z}^3}.
\end{array}
$$

where $X = -Z^0\partial/\partial Z^0 + Z^1\partial/\partial Z^1 = -\hat{Z}^0\partial/\partial\hat{Z}^0 + \hat{Z}^1\partial/\partial\hat{Z}^1$ (the generator one of the symmetries). The corresponding transition matrices are $P_{20} = \mathrm{diag}(1, w^{-1}, w^{-1}, 1)$, $P_{31} = \mathrm{diag}(w^{-1}, 1, 1, w^{-1})$, and

$$
P_{01} = \begin{pmatrix} g(w) & 1 & wg(w) & wg(w) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

where $g(w) = (q + w)/w(Q + w)$, with $q = \frac{1}{2}(Q + a^2)$. We interpret w as the coordinate on the non-Hausdorff reduced twistor space.

Note that the vectors parallel to $Z^\alpha\partial/\partial Z^\alpha$ span a line sub-bundle of E is isomorphic to L^{-1}, and that $F := E/L^{-1} = T\mathbb{P}\mathcal{T} \otimes L^{-1}$.

The general anti-self-dual vacuum metric with two commuting orthogonally transitive Killing vectors is

$$
\mathrm{d}s^2 = f(\mathrm{d}t - \omega\mathrm{d}\theta)^2 + f^{-1}(\mathrm{d}r^2 + \mathrm{d}z^2 + r^2\mathrm{d}\theta^2)\ (*)
$$

where ω and f are functions of r and z such that $f^2\omega_z = rf_r$ and $f^2\omega_r = -rf_z$. By considering local twistor transport in such a background, one can show that the bundle $E \otimes L$ always has transition matrices of this form, where in the general case $g(z) = f(0, z)^{-1}$. The top left-hand two-by-two block in P_{01}

is the patching matrix P, which characterizes the solution uniquely. We can immediately read off, therefore, that the type D solution in the preceding article is given by (*), with

$$\frac{1}{f(0,z)} = \frac{q}{Qz} + \frac{p}{Q(Q+z)}.$$

Since $f^{-1}(r,z)$ is an axisymmetric harmonic function in cylindrical polars, we conclude that it is the potential of a pair of point masses p/Q and q/Q separated by Q.

A vacuum solution of the form (*) is type D if it admits a non-null Killing spinor ω^{AB} (this is a nontrivial condition, although there are many Killing spinors with primed indices whatever the form of the curvature). A Killing spinor with unprimed indices determines a holomorphic section A of the symmetric tensor product $F \otimes_S F = L^{-2} \otimes (T\mathbb{P}T \otimes_S T\mathbb{P}T)$. By dropping the fourth element of the local frame for E in V_0 and V_2, and the third element in V_1 and V_3, we can represent $L \otimes F = T\mathbb{P}T$ by the transition matrices $M_{20} = \mathrm{diag}(1, w^{-1}, w^{-1})$, $M_{31} = \mathrm{diag}(w^{-1}, 1, w^{-1})$, and

$$M_{01} = \begin{pmatrix} g(w) & 1 & -wg(w) \\ 1 & 0 & -2w \\ 0 & 0 & 1 \end{pmatrix}$$

A section of $F \otimes_S F$ has local representatives $A_i(w)$, where $i = 0, 1, 2, 3$ and the As are symmetric 3×3 matrices. For a global section

$$A_2 = wM_{20}A_0M_{20}^t, \quad A_3 = wM_{31}A_1M_{31}^t, \quad A_0 = M_{01}A_1M_{01}^t,$$

with A_2 and A_3 well behaved near $w = \infty$, and A_0 and A_1 well behaved near $w = 0$. The first two transition relations imply that the dependence of A_0 and A_1 on w must be of the form

$$A_0 = \begin{pmatrix} 0 & C & C \\ C & L & L \\ C & L & L \end{pmatrix}, \quad A_1 = \begin{pmatrix} L & C & L \\ C & 0 & C \\ L & C & L \end{pmatrix}$$

where C means 'constant' and L means 'linear'. After a little algebra, the third relation then implies that g must be of the form $g = g_1(w)/g_2(w)$, where g_1 is at most linear in w and g_2 is at most quadratic. For the solution in the previous article, $g_1 = q - pw$ and $g_2 = Qw + w^2$: this is essentially the generic case. For the 'Euclidean Taub-NUT' solution, $g_1 = w$ and $g_2 = w + 2m$.

[1] Tod, K. P. and Ward, R. S. (1979) Self-dual metrics with self-dual Killing vectors *Proc. Roy. Soc. Lond.* **A368** 411–27

[2] Ward, R. S. (1983) Stationary axisymmetric space-times: a new approach *Gen. Rel. Grav.* **15** 105–9

[3] Woodhouse, N. M. J. and Mason, L. J. (1988) The Geroch group and non-Hausdorff twistor space *Nonlinearity* **1** 73–114

[4] Fletcher, J. and Woodhouse, N. M. J. (1990) Twistor characterization of stationary axisymmetric solutions of Einstein's equations. In *Twistors in mathematics and physics*, eds. T. N. Bailey and R. J. Baston. LMS Lecture Note Series **156**, Cambridge University Press.

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