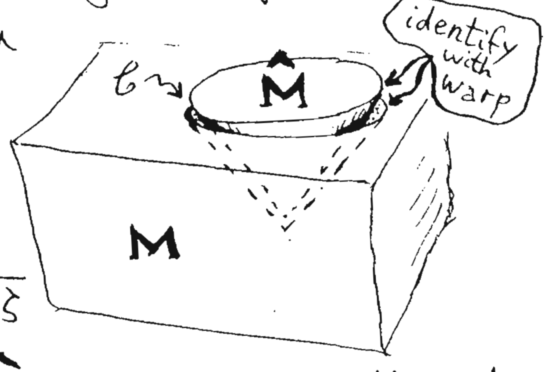
On Impulsive Gravitational Waves

The general pure δ-function solution of the Einstein vacuum equations was described in Penrose (1972), using a "scissors and paste" construction. Such waves can have plane or spherical wave fronts, the former being limiting cases of the latter. We discuss only the spherical case here. The scissors and paste description can be given by matching a region $\mathbf{M}$ of Minkowski space outside a (say future) light cone $\mathcal{C}$, with metric



$$ds^2 = 2\,du\,dv - 2u^2 d\zeta d\bar{\zeta}$$

to another Minkowskian region $\hat{\mathbf{M}}$ inside $\mathcal{C}$, with metric

$$ds^2 = 2\,d\hat{u}\,d\hat{v} - 2\hat{u}^2 d\hat{\zeta} d\bar{\hat{\zeta}},$$

where the metric identification ("warp") at $\mathcal{C}$ is given by

$$\hat{v} = 0 = v$$
$$\hat{\zeta} = f(\zeta) \qquad\qquad * $$
$$\hat{u} = u/|f'(\zeta)|,$$

f being holomorphic, apart from at its singular regions, which may be thought of as singular "wires" on $\mathcal{C}$. The Einstein vacuum equations are satisfied across $\mathcal{C}$, along which there is a δ-function in the Weyl curvature.

It is of interest to note that the (u, ζ) transformation can be described neatly as a non-linear holomorphic transformation of the spin-space, at the vertex $\mathbf{O}$ of $\mathcal{C}$, preserving

$\boxed{\xi_A\, d\xi^A}$ i.e. both of $\boxed{d\xi_A \wedge d\xi^A}$ and $\boxed{\xi^A \frac{\partial}{\partial \xi^A}}$.

Here, the position vector of a point on the cone is given by $\xi^A \bar{\xi}^{A'}$ $(= x^a)$, which in coordinates is $(\frac{1}{\sqrt{2}} \times)$

$$\begin{bmatrix} u & \bar{\zeta}u \\ \zeta u & \zeta\bar{\zeta}u \end{bmatrix} = \begin{bmatrix} \xi^0\bar{\xi}^{0'} & \xi^0\bar{\xi}^{1'} \\ \xi^1\bar{\xi}^{0'} & \xi^1\bar{\xi}^{1'} \end{bmatrix},$$

so $*$ becomes $\xi^A \mapsto \hat{\xi}^A$ according to

$$\xi^0 \mapsto \hat{\xi}^0 = \xi^0 (f'(\zeta))^{-1/2}$$
$$\xi^1 \mapsto \hat{\xi}^1 = \xi^0 f(\zeta)(f'(\zeta))^{-1/2}$$

where

$$\zeta = \xi^1/\xi^0$$

i.e.

$$\zeta \mapsto \hat{\zeta} = f(\zeta) \qquad\qquad \text{where } \begin{pmatrix}\xi^0\\ \xi^1\end{pmatrix} = \begin{pmatrix}\eta^{1/2}\\ \zeta\eta^{1/2}\end{pmatrix}.$$
$$\eta \mapsto \hat{\eta} = \eta/f'(\zeta)$$

This preserves

$$\xi_A d\xi^A = \xi^0 d\xi^1 - \xi^1 d\xi^0 = (\xi^0)^2 d\left(\frac{\xi^1}{\xi^0}\right) = \eta\, d\zeta.$$

This fact is closely related to the Hamiltonian nature of the twistor transformation between $\mathbf{M}$'s and $\hat{\mathbf{M}}$'s twistor spaces, that was noted in Penrose & MacCallum (1972).

The metric on the entire space $\mathcal{M} = \mathbf{M} \cup \mathcal{C} \cup \hat{\mathbf{M}}$ can be described as a $\mathbf{C}^0$ metric form

$$ds^2 = 2\,du\,dv - 2\left|u\,d\bar{\zeta} + v\{h;\zeta\}\,d\zeta\right|^2$$

where $\{\,;\,\}$ stands for the Schwarzian derivative

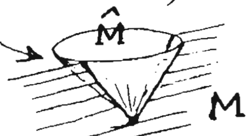
$$\{h;\zeta\} = -\frac{1}{2}\left(\frac{h_{\zeta\zeta\zeta}}{h_\zeta} - \frac{3}{2}\left(\frac{h_{\zeta\zeta}}{h_\zeta{}^2}\right)^2\right).$$

The curvature is defined by the only surviving Weyl component

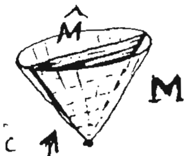
$$\Psi_4 = \frac{1}{u}\{h;\zeta\}_\zeta\,\delta(v)$$

in a suitable spin frame (o^A pointing along generators of $\mathcal{C}$). (Nutku 1990.)

A simple example is given by a "snapping cosmic string" (also described, in a different C^0 way by Gleiser & Pullin 1989), where $f(\zeta) = \zeta^{1+\varepsilon}$. Here, the "wires" at the north and south poles arise from a deficit-angle identification for a cosmic string in $\mathbf{M}$ which snaps, in $\mathcal{M}$, emitting a gravitational wave along $\mathcal{C}$

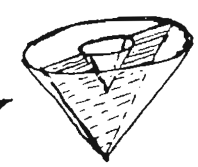


— or else in $\hat{\mathbf{M}}$, **where** here the cosmic string is <u>created</u> by the gravitational wave along $\mathcal{C}$

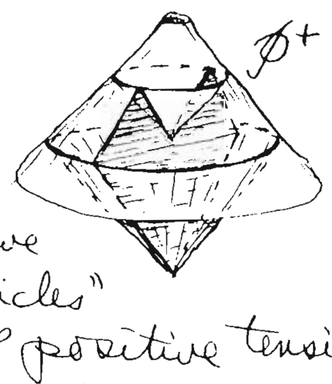


— or, analogously, we could snap or create a so-called "rotating" cosmic string if we allow ε to be complex.

An interesting consideration is that of energy balance, e.g. in the situation

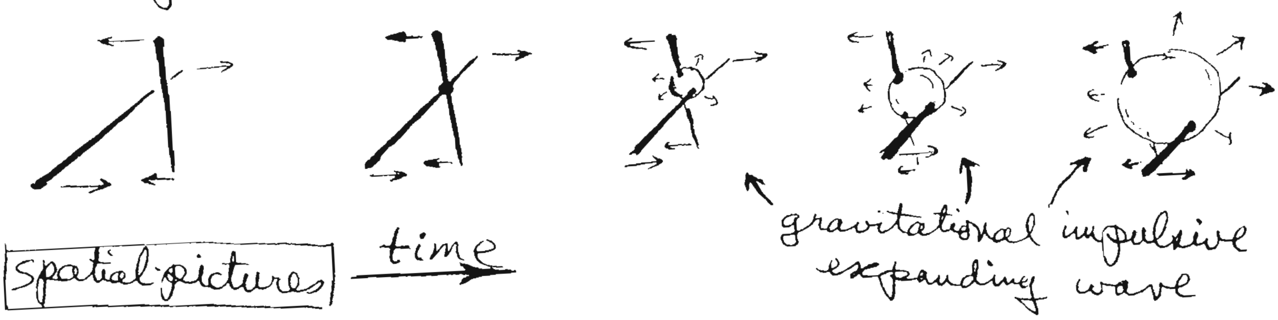


, where two string segments separate with the speed of light, having previously been joined as one string that was created with the emission of the first of the two gravitational wave bursts. Here the gravitational energy in the waves is infinite owing to an angular divergence, though the time-integral of energy flux along each generator of $\mathscr{I}^+$ is proportional to the length of the string segments.

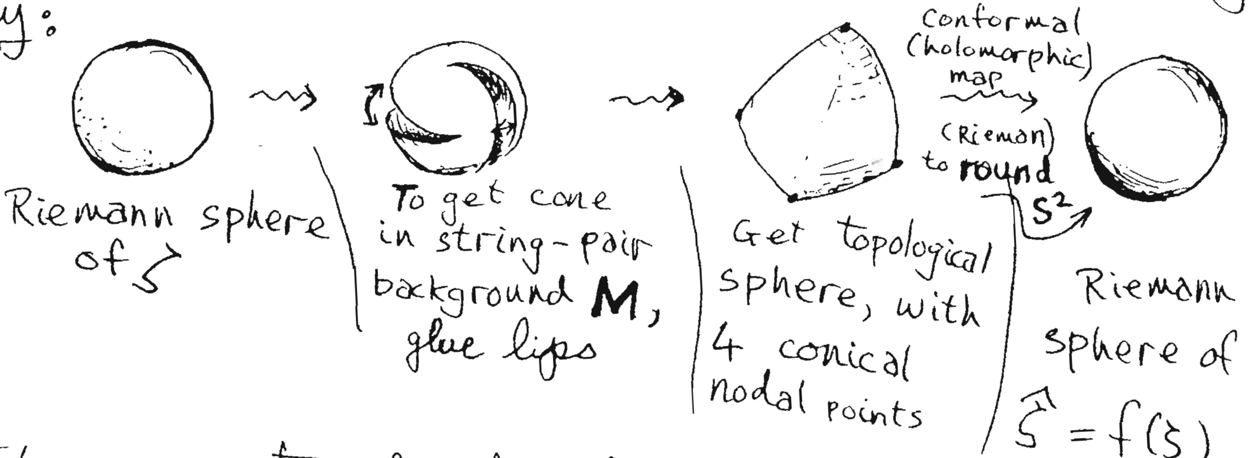


To make sense of all this, we must observe that the string segments have "particles" at their ends, where for a string of positive tension (and positive mass) the leading particle has a negative mass that decreases (becomes more negative) with time and the trailing one has a positive mass that increases with time; for a string of negative tension (i.e. positive pressure and hence negative mass) it is the other way around. (All the masses for these "particles" are inertial masses; they have zero rest-mass.)

A more involved situation is provided by a pair of strings that collide



Here the function h is obtained in the following way:



It seems to be hard to find f explicitly for this case, but a Riemann theorem ensures that f exists.

References

Penrose, R. (1972) The Geometry of Impulsive Gravitational Waves in General Relativity (papers in honour of J.L. Synge) (Clarendon Press, Oxford) 101–115.

Penrose, R. & MacCallum, M.A.H. (1972) Twistor theory: an approach to the quantization of fields and space-times Phys. Rept. **6C**, 241–315

Gleiser, R. & Pullin, J. (1989) Are cosmic strings stable topological defects? Class. Quantum Grav. **6**, L141–L144.

Nutku, Y. (1991) Spherical shock waves in general relativity Phys. Rev. D. **44**, 3164–3168

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