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Kinking and Causality

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Introduction

Recently, there has been some speculation along the following lines:

Suppose M is a compact spacetime, with

$$\partial M \cong \Sigma \ncong \emptyset$$

(Σ may be single three-manifold or the disjoint union of several). Let v be a timelike vector field with respect to the Lorentz metric, and let $\mathrm{kink}(\partial M; v)$ denote the kinking number of v with respect to ∂M (see [1] or [2]).

Recently, there has been some suspicion that there may be a relation between the topology of ∂M, along with the value of $\mathrm{kink}(\partial M; v)$, and the existence of closed timelike curves in M. In particular it has been conjectured that if $\partial M \cong S^3$ and $\mathrm{kink}(\partial M; v) = 0$, then there must exist closed timelike curves in M (M assumed to be space and time orientable).

In this paper, we show that the above conjecture is false (by counterexample). In fact, we prove the more general

Proposition 1 Let Σ be any closed, orientable three-manifold, $n \in \mathbb{Z}$ an arbitrary integer. Then there exists a compact causal spacetime M with $\partial M \cong \Sigma$ and $\mathrm{kink}(\partial M; v) = n$, where v is a timelike vector field.

Proposition 2 If M is compact and causality violating, with $\partial M \cong \Sigma \ncong \emptyset$, then there exists a continuous deformation of the metric on M such that the new spacetime with deformed metric does not possess closed timelike curves.

(Note: Deforming the metric does not alter the kinking number).

The proofs of Propositions 1 and 2 draw on the idea of the counterexample.

Construction of Counterexample

To construct the example, consider the manifold

$$M \cong \mathbb{CP}^2 \,\#\, (S^1 \times S^3) \tag{1}$$

where $\#$ denotes "connected sum". Let $e(M)$ = "Euler number of M", then $e(M) = e(\mathbb{CP}^2) + e(S^1 \times S^3) - 2$.

Since $e(\mathbb{CP}^2) = 3$ and $e(S^1 \times S^3) = 0$ we find

$$e(M) = 1 \tag{2}$$

Define the manifold M' by

$$M' \cong M - D^4 \ , \tag{3}$$

where D^4 is a four-ball. Then

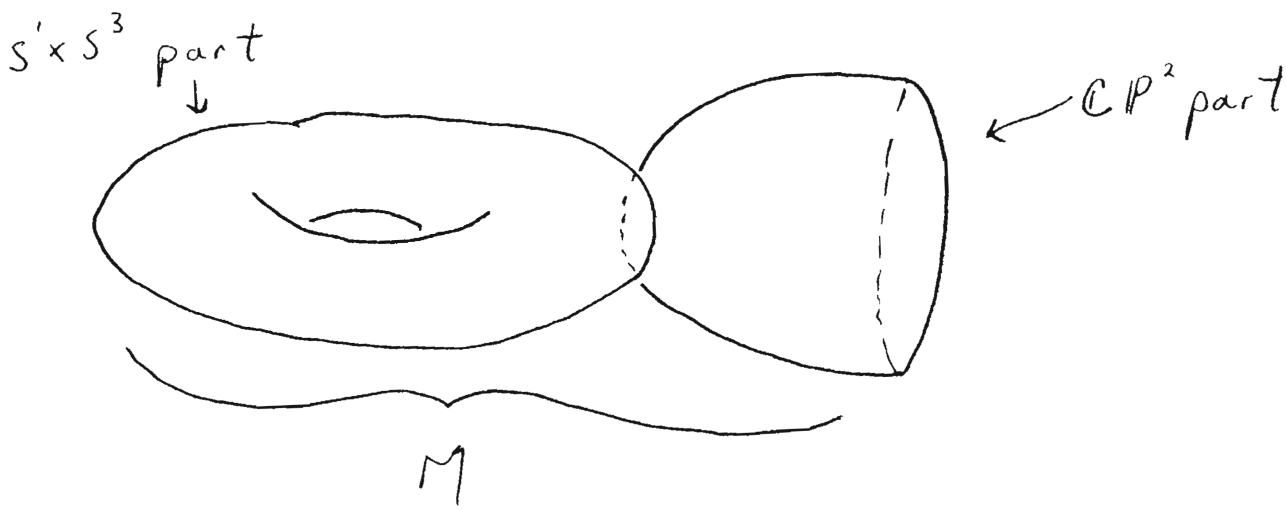
$$e(M') = 0 \tag{4}$$

Thus, we can put a nonvanishing vector field v on M' which has zero kinking on $\partial M' \cong S^3$, i.e.,

$$\text{kink}(\partial M'; v) = 0 \tag{5}$$

Now, one may suppose that there are closed timelike curves in M'; in fact, if v is outward normal on $\partial M'$ there must exist closed timelike curves (by a standard argument). However, we shall now show that we can always "cut" all of the closed timelike curves in $M' \cong M - D^4$ by choosing the D^4 that we remove from M cleverly. We shall do this "choosing" in an essentially constructive manner.

Hence, take $M \cong \mathbb{CP}^2 \# (S^1 \times S^3)$ as above and let v be a vector field on M, i.e., visually:



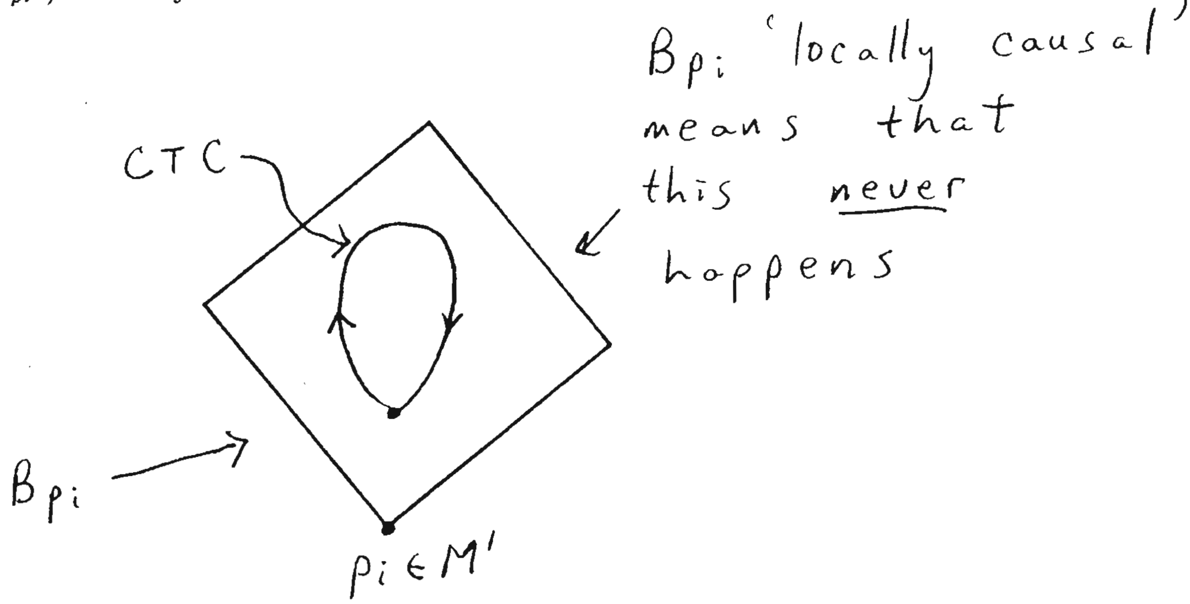
Remove a ball D^4 from around the singular point of v, so that

$$\partial D^4 \cong \partial M' \cong S^3.$$

Now, we can cover M' with a finite number of sets B_{pi} of the form

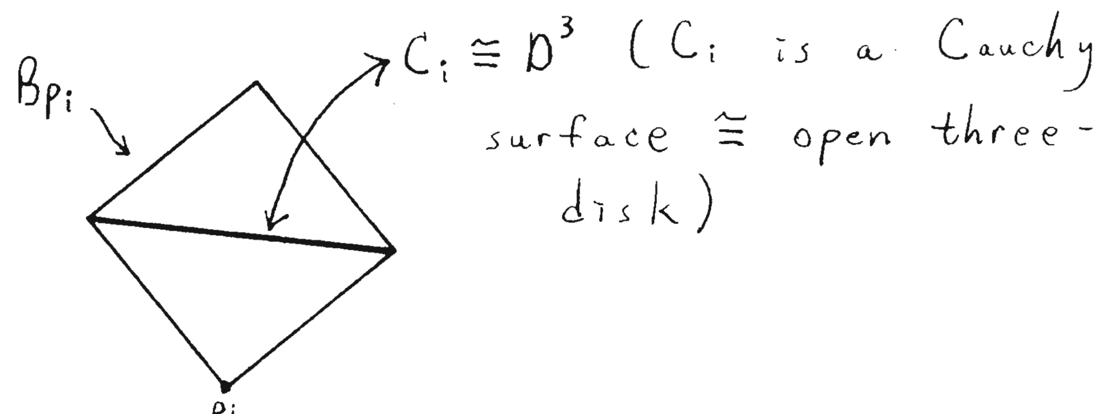
$$B_{pi} = \{x \in I^+(p_i) \cap I^-(q) \mid q \in I^+(p_i)\}$$

Furthermore, we can take the sets in this finite cover to be fine enough that they are all locally causal (i.e., no closed timelike curve, or CTC, lies entirely in any one of the B_{pi}s). Visually:

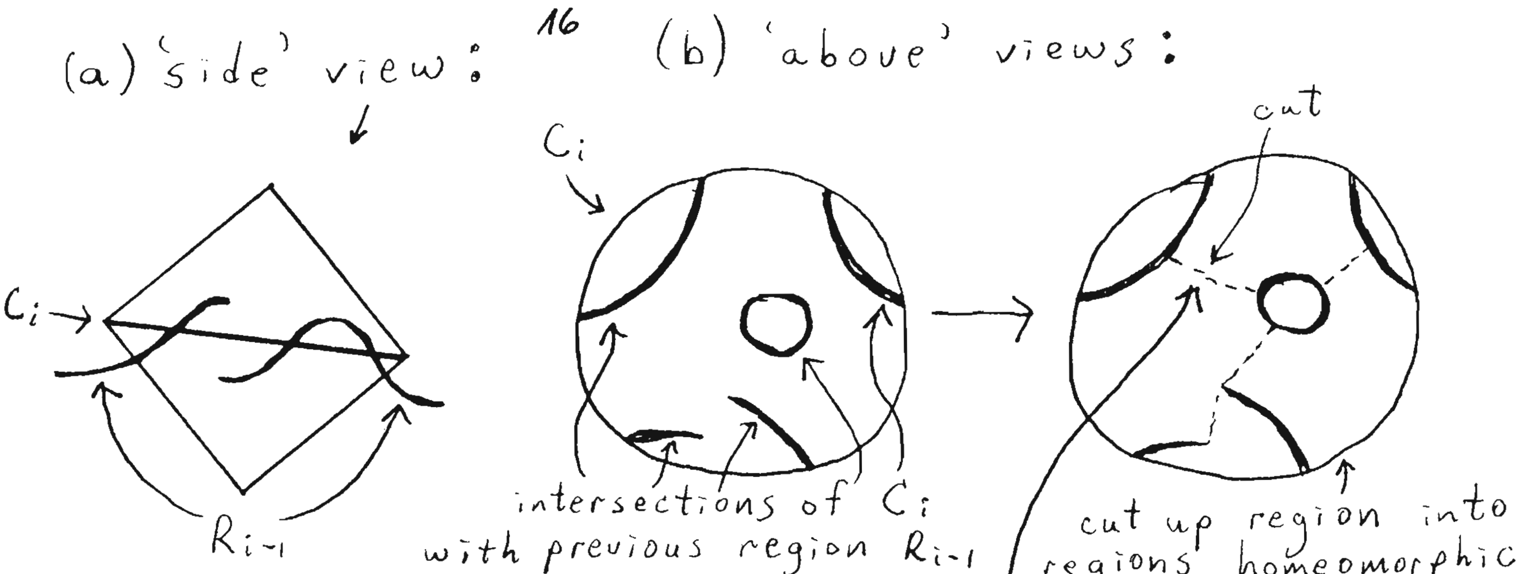


Now, the crucial idea of the construction depends upon our ability to cut all of the CTCs by removing a finite number of four - balls. That we can do this is reasonably intuitively obvious, but we justify this construction more rigorously as follows.

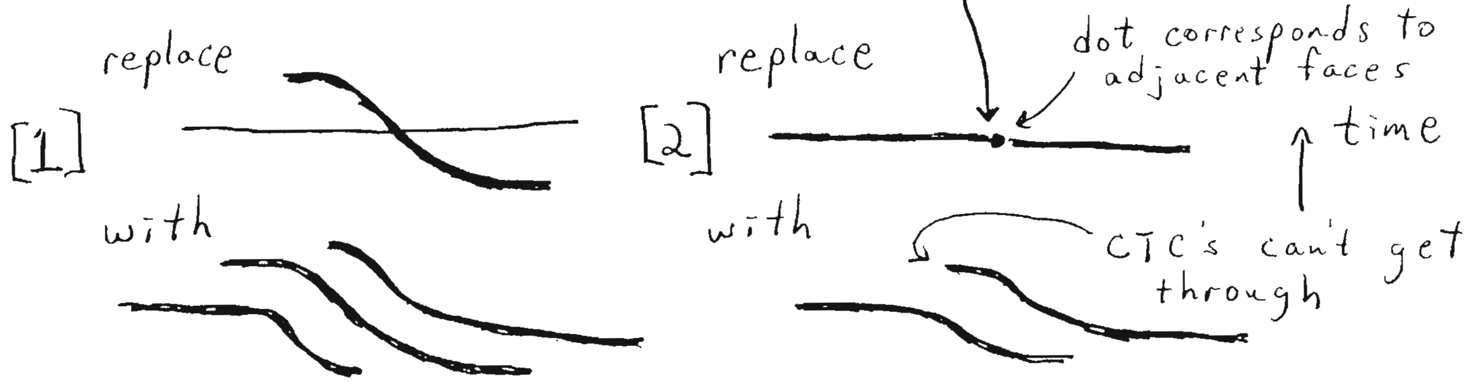
Begin by successively removing the '$t = 0$' Cauchy surface, C_i from each of our locally causal covering sets B_{pi}, as shown:



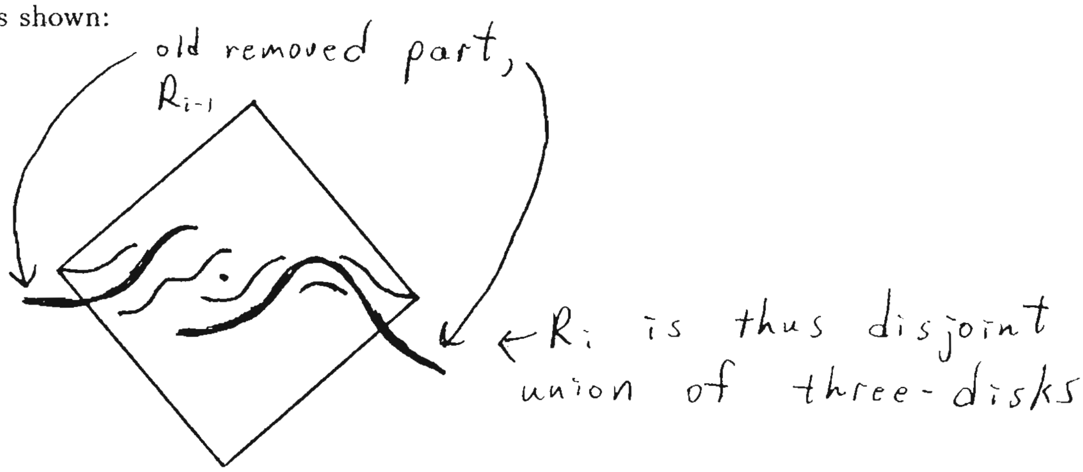
Now, at each stage C_i may already be intersected by a previously removed part (assumed to be a union of three-disks), R_{i-1}, so subdivide to get a covering of what's left by three-disks, (D^3s), as shown:



Next, modify C_i according to the following two rules:

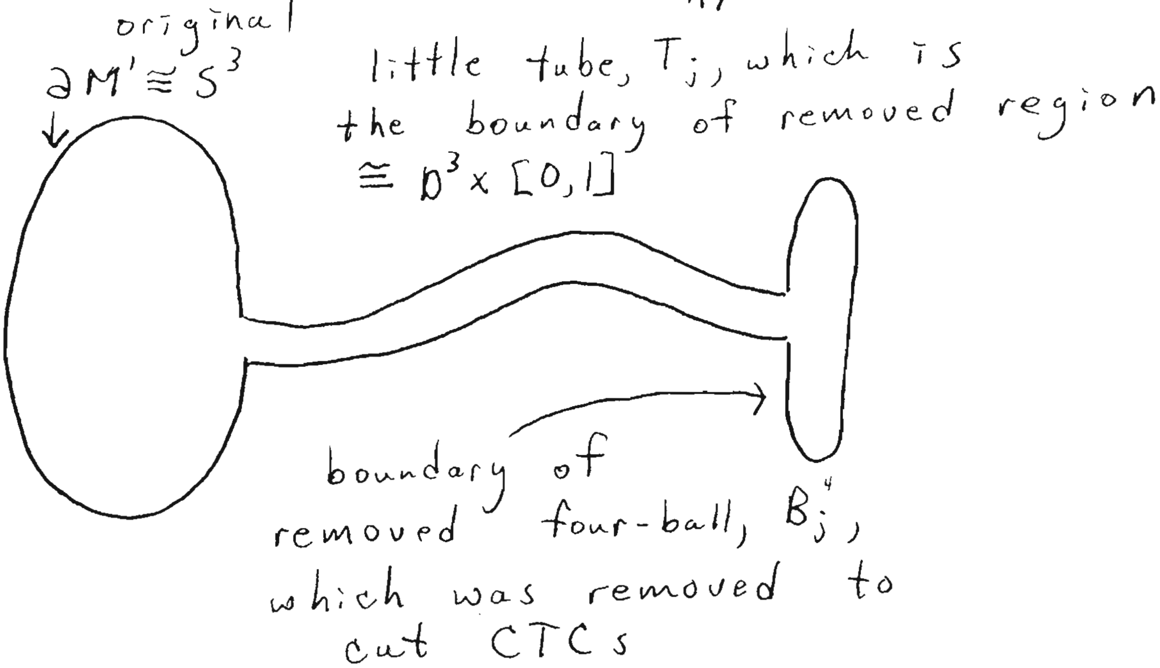


Adjoin the result to R_{i-1} to get R_i, which is thus given as a disjoint union of three-balls, D^3_j, as shown:



Finally, thicken out the D^3_js to get disjoint four-balls B^4_js which clearly cut the CTCs.

Hence, we can cut all of the CTC's with a finite number of such four - balls. We now connect each of these 'cut out regions' to the original deleted region (i.e., where D^4 was) via 'little tubes' $T_j \cong S^3 \times [0,1]$; that is, we cut out a little tube leading from the old boundary of M' ($\partial M' \cong \partial D^4$) to the new boundary component formed by removing B^4_j, as shown:



Call the new manifold obtained after such a finite sequence of operations 'N'. Then clearly

$$\partial N \cong S^3$$

since the total topology of the removed regions

$$R \cong D^4 \cup T_1 \cup T_2 \cup ... \cup T_n \cup B_1^4 \cup B_2^4 \cup ... \cup B_n^4$$

is still D^4, and $\partial D^4 \cong S^3$. Furthermore, v is still global and nonvanishing on N, and $e(N) = 0$; hence, $\text{kink}(\partial N; v) = 0$.

Thus, N is a causal spacetime (which is orientable) for which $\partial N \cong S^3$ and $\text{kink}(\partial N; v) = 0$; hence, N constitutes a counterexample to the conjecture mentioned in the introduction.

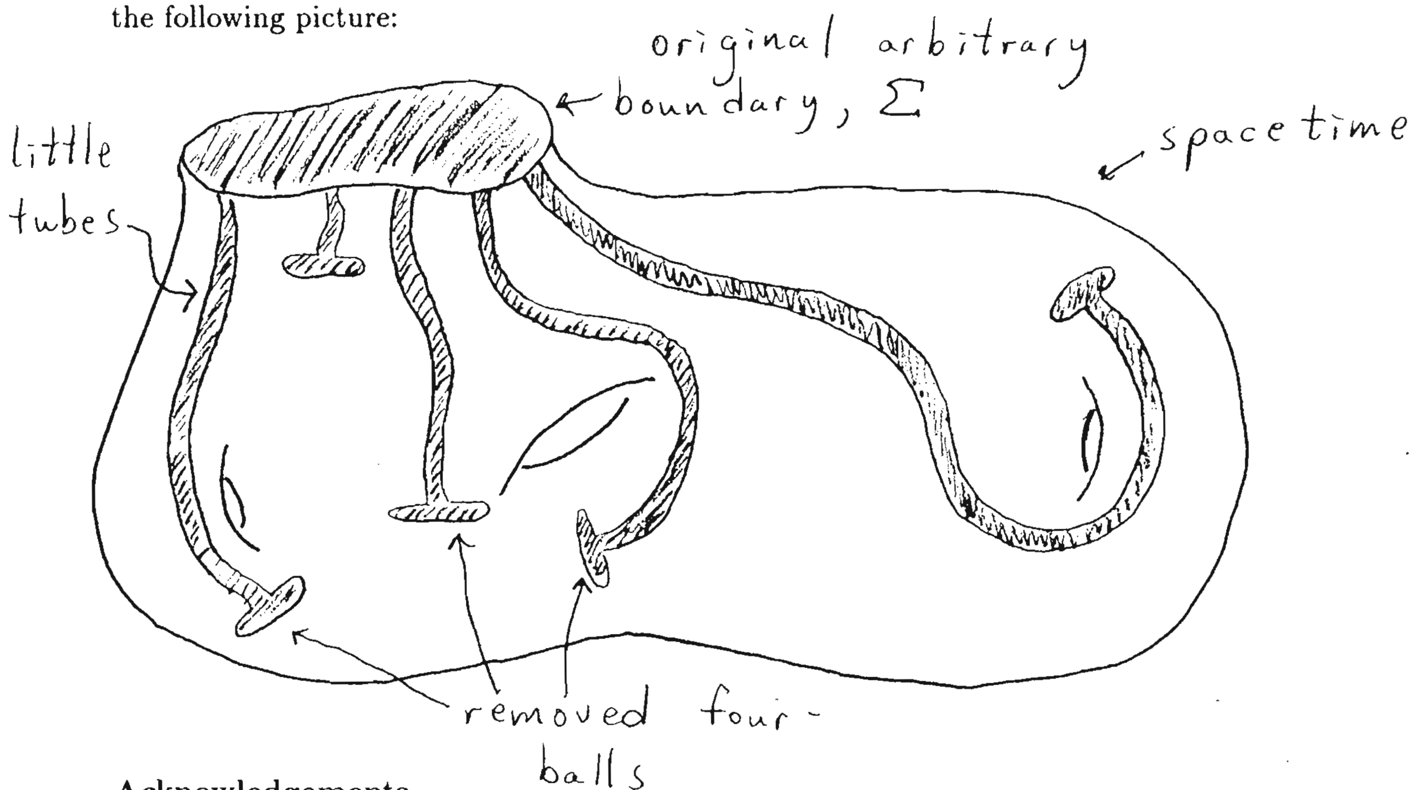
Proof of the general statement

To prove the more general Propositions (stated in the introduction) we simply generalize the above construction.

That is, let Σ be any three-manifold (or perhaps disjoint union of three-manifolds) and $n \in \mathbb{Z}$ any integer. Then we can always find a Lorentz manifold M (with timelike vector v) such that $\partial M \cong \Sigma$ and $\text{kink}(\partial M; v) = n$. This follows from the general formula

$$e(M) = \Sigma i_v \ + \ \text{kink}(\partial M; v) \tag{6}$$

(see [2]). If M should happen to possess CTC's we can always do the above construction and "cut" them by removing a finite number of four - balls B_j^4 and connecting these four - balls to the original boundary by removing little (nonintersecting) tubes T_j. (For Proposition 2, we simply continuously retract the T_js and the B_j^4s, dragging the metric with them). The fundamental idea of this paper, then, is represented in the following picture:



Acknowledgements

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References

[1] G.W. Gibbons and S.W. Hawking, *Kinks and Topology Change,* DAMTP preprint

[2] H.A. Chamblin, *M.Sc. Thesis*, University of Oxford (1992), In process of completion